AIM :

* Linear functional :-

Let V be a vectorspace mapping $f: V \to R$ is called linear functional
if (1) $f(x+y) = f(x) + f(y)$
(2) $f(\alpha x) = \alpha f(x)$
or $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$
i.e A Linear functional is a Linear Transformation mapping $T: V \to \mathbb{R}$.

* Dual space (L(U, R) space) :

The set (vectorspace) $L(U, R)$ of all linear functionals on U is called Dual space of U and it is denoted by U'

* Dual Basis :-

If U is n-dimensional vectorspace and $B_1 = \{x_1, x_2, \dots, x_n\}$ be a basis of U, R is 1-dimensional and $B_2 = \{1\}$ is a basis of $\mathbb{R}$ then $B_3 = \{f_{1j} / 1 \le j \le n\}$ becomes ordered basis for $L(U, \mathbb{R})$ where f_{1j} are linear functionals defined as follows.

$$f_{1j}(x_j) = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \ne j \end{cases}$$

AIM : finite,

Q-1

(1) Consider basis $B_1 = \{x_1 = (-1, 2), x_2 = (5, 3)\}$ of $\mathbb{R}^2$.

Sol^n:- $U = \mathbb{R}^2$

$L(\mathbb{R}^2, \mathbb{R}) : \{f \mid f : \mathbb{R}^2 \to \mathbb{R}\}$

Let $(a_1, a_2) \in \mathbb{R}^2$

$$f_{11}(a_1, a_2) = a_1 b_1 + a_2 b_2$$
$$f_{12}(a_1, a_2) = a_1 c_1 + a_2 c_2$$

$f_{11}(x_1) = 1$

$f_{11}(-1, 2) = -b_1 + 2b_2 = 1 \quad -(1)$

$f_{11}(x_2) = 0$

$f_{11}(x_2) = 5b_1 + 3b_2 = 0 \quad -(2)$

$5(1) + (2)$

$$-5b_1 + 10b_2 = 5$$
$$5b_1 + 3b_2 = 0$$
$$13b_2 = 5$$

$$\therefore b_2 = \frac{5}{13}$$

Teacher's Signature :

Expt. No. ______

Date ______

AIM :

Let us put the value of b_2 in (1)

$$-b_1 + 2\left(\frac{5}{13}\right) = 1$$

$$-b_1 + \frac{10}{13} = 1$$

$$\therefore \frac{10}{13} - 1 = b_1$$

$$\therefore b_1 = \frac{-3}{13}$$

$$f_{11}(a_1, a_2) = \frac{-3a_1}{13} + \frac{5a_2}{13}$$

→ $f_{12}(x_1) = 0$

$f_{12}(-1, 2) = -c_1 + 2c_2 = 0$ — (3)

$f_{12}(x_2) = 1$

$f_{12}(5, 3) = 5c_1 + 3c_2 = 1$ — (4)

$5(3) + (4)$

$$-5c_1 + 10c_2 = 0$$
$$5c_1 + 3c_2 = 1$$
$$\therefore 13c_2 = 1$$

Teacher's Signature : ______

Expt. No. ____ Date ____

AIM :

$$C_2 = \frac{1}{13}$$

Let us put the value of C_2 in (3)

$$-c_1 + 2\left(\frac{1}{13}\right) = 0$$

$$-c_1 + \frac{2}{13} = 0$$

$$\therefore \quad c_1 = \frac{2}{13}$$

$$f_{12}(a_1, a_2) = \frac{2a_1}{13} + \frac{a_2}{13}$$

Hence the dual basis

$$L(\mathbb{R}^2, \mathbb{R}) = \left\{ f_{11}(a_1, a_2) = \frac{-3a_1}{13} + \frac{5a_2}{13}, \right.$$

$$\left. f_{12}(a_1, a_2) = \frac{2a_1}{13} + \frac{a_2}{13} \right\}$$

Q - 2

(2) Consider basis $B_1 = \{x_1 = (2,3), x_2 = (-5,1)\}$ of $\mathbb{R}^2$.

Sol^n:- $U = \mathbb{R}^2$

$L(\mathbb{R}^2, \mathbb{R}) = \{f \mid f : \mathbb{R}^2 \to \mathbb{R}\}$

Let $(a_1, a_2) \in \mathbb{R}^2$

$f_{11}(a_1, a_2) = a_1 b_1 + a_2 b_2$
$f_{12}(a_1, a_2) = a_1 c_1 + a_2 c_2$

$f_{11}(x_1) = 1$
$f_{11}(2,3) = 2b_1 + 3b_2 = 1 \quad -(1)$

$f_{11}(x_2) = 0$
$f_{11}(-5,1) = -5b_1 + b_2 = 0 \quad -(2)$

$(1) - 3(b_2 2)$

$$\begin{aligned} 2b_1 + 3b_2 &= 1 \\ -15b_1 + 3b_2 &= 0 \\ + \quad - \quad & \quad - \\ \hline 17 b_1 &= 1 \end{aligned}$$

$$b_1 = \frac{1}{17}$$

Expt. No.

AIM :

Let us put the value of b_1 in eq^n (2)

$$-5\left(\frac{1}{17}\right) + b_2 = 0$$

$$\frac{-5}{17} + b_2 = 0$$

$$b_2 = \frac{5}{17}$$

$$f_{11}(a_1, a_2) = \frac{a_1}{17} + \frac{5a_2}{17}$$

$$f_{12}(x_1) = 0$$

$$f_{12}(2, 3) = 2c_1 + 3c_2 = 0 \quad -(3)$$

$$f_{12}(x_2) = 1$$

$$f_{12}(-5, 1) = -5c_1 + c_2 = 1 \quad -(4)$$

(3) - 3(4)

$$2c_1 + 3c_2 = 0$$

$$-15c_1 + 3c_2 = 3$$

$$+ \quad - \quad -$$

$$17c_1 = -3$$

$$c_1 = -3/17$$

Teacher's Signature :

VISION

Date

Let us put the value of c_1 in (3)

$$2\left(\frac{-3}{17}\right) + 3c_2 = 0$$

$$\frac{-6}{17} + 3c_2 = 0$$

$$3c_2 = \frac{6}{17}$$

$$\therefore \quad c_2 = \frac{2}{17}$$

$$f_{12}(a_1, a_2) = \frac{-3a_1}{17} + \frac{2a_2}{17}$$

Hence the dual basis

$$L(\mathbb{R}^2, \mathbb{R}) = \left\{ f_{11}(a_1, a_2) = \frac{a_1}{17} + \frac{5a_2}{17}, \quad f_{12}(a_1, a_2) = \frac{-3a_1}{17} + \frac{2a_2}{17} \right\}$$

Expt. No.

AIM :

Q-3

(3) $B_2 = \{ x_1 = (1,0,0), x_2 = (1,1,0), x_3 = (1,1,1$

Sol^n: Let $V = \mathbb{R}^3$

$L(\mathbb{R}^3, \mathbb{R}) = \{ f \mid f : \mathbb{R}^3 \to \mathbb{R} \}$

Let $(a_1, a_2, a_3) \in \mathbb{R}^3$

$f_{11}(a_1, a_2, a_3) = a_1 b_1 + a_2 b_2 + a_3 b_3$
$f_{12}(a_1, a_2, a_3) = a_1 c_1 + a_2 c_2 + a_3 c_3$
$f_{13}(a_1, a_2, a_3) = a_1 d_1 + a_2 d_2 + a_3 d_3$

$f_{11}(x_1) = 1$
$f_{11}(1,0,0) = b_1 = 1 \quad -(1)$

$f_{11}(x_2) = 0$
$f_{11}(1,1,0) = b_1 + b_2 = 0 \quad -(2)$

$f_{11}(x_3) = 0$
$f_{11}(1,1,1) = b_1 + b_2 + b_3 = 0 \quad -(3)$

from (1) & (2)

$1 + b_2 = 0$

$\therefore b_2 = -1 \quad -(4)$

AIM :

from (1), (3) & (4)

$$1 - 1 + b_3 = 0$$

$$\therefore b_3 = 0$$

$f_{11}(a_1, a_2, a_3) = a_1 - a_2$

$f_{12}(x_1) = 0$

$f_{12}(1, 0, 0) = c_1 = 0 \quad -(5)$

$f_{12}(x_2) = 1$

$f_{12}(1, 1, 0) = c_1 + c_2 = 1 \quad -(6)$

$f_{12}(x_3) = 0$

$f_{12}(1, 1, 1) = c_1 + c_2 + c_3 = 0 \quad -(7)$

from (5) & (6)

$0 + c_2 = 1 \quad \therefore c_2 = 1 \quad -(8)$

from (5), (7) & (8)

$$0 + 1 + c_3 = 0$$

$$\therefore c_3 = -1$$

Expt. No.

AIM :

$$f_{12}(a_1, a_2, a_3) = a_2 - a_3$$

$f_{13}(x_1) = 0$

$f_{13}(1,0,0) = d_1 = 0 \quad -(9)$

$f_{13}(x_2) = 0$

$f_{13}(1,1,0) = d_1 + d_2 = 0 \quad -(10)$

$f_{13}(x_3) = 1$

$f_{13}(1,1,1) = d_1 + d_2 + d_3 = 1 \quad -(11)$

from (9) & (10)

$\therefore 0 + d_2 = 0$

$\therefore d_2 = 0 \quad -(12)$

from (9), (11), (12)

$0 + 0 + d_3 = 1$

$\therefore d_3 = 1$

$$f_{13}(a_1, a_2, a_3) = a_3$$

Hence the dual basis

$L(\mathbb{R}^3, \mathbb{R}) = \{ f_{11} = a_1 - a_2,\ f_{12} = a_2 - a_3,\ f_{13} = a_3 \}$

AIM :

Date

Hence, the dual basis

$$L(\mathbb{R}^3, \mathbb{R}) = \left\{ \begin{array}{l} f_{11}(a_1, a_2, a_3) = a_1 - a_2, \\ f_{12}(a_1, a_2, a_3) = a_2 - a_3, \\ f_{13}(a_1, a_2, a_3) = a_3 \end{array} \right\}$$

Page No.
Date
Expt. No.

AIM :

* Dual basis existence theorem :

Statement :

Let V be a finite dimensional vectorspace over a field F. For every basis $v_1, v_2, \ldots, v_n$ of V there is a basis $f_1, f_2, \ldots, f_n$ of V' such that

$$f_i(v_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Proof :-

Define f_i as follows.

For $v \in V$ we set $f_i(v) = \alpha_i$ where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ are such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

$$V = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

$$f_i(v) = f_i(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$$

$$f_i(v) = \alpha_1 f_i(v_1) + \alpha_2 f_i(v_2) + \cdots + \alpha_n f_i(v_n)$$

$$f_i(v) = \alpha_i$$

VISION

Teacher's Signature :

Date

AIM :

We want to prove that f_i is L.T

i.e. $f_i \in L(V, \mathbb{R})$

Let $w \in V$

$w = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$

Let $k_1, k_2 \in F$ then, $v = \sum_{j=1}^{n} \alpha_j v_j$

$$f_i(k_1 v + k_2 w) = f_i\left(k_1 \sum_{j=1}^{n} \alpha_j v_j + k_2 \sum_{j=1}^{n} \beta_j v_j\right)$$

$$f_i(k_1 v + k_2 w) = f_i\left(\sum_{j=1}^{n} (k_1 \alpha_j v_j + k_2 \beta_j v_j)\right)$$

$$f_i(k_1 v + k_2 w) = f_i\left(\sum_{j=1}^{n} (k_1 \alpha_j + k_2 \beta_j) v_j\right)$$

$$f_i(k_1 v + k_2 w) = k_1 \alpha_i + k_2 \beta_i$$

$(\because i = j$

$\therefore f_i(v_j) = 1)$

$(\because f_i(v) = \alpha_i$

$f_i(w) = \beta_i)$

$$f_i(k_1 v + k_2 w) = k_1 f_i(v) + k_2 f_i(w)$$

$\therefore f_i \in V'$

Teacher's Signature :

Expt. No. ______ Date ______

AIM :

We have thus found elements $f_1, f_2, \ldots f_n$ of V' such that,

$$f_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Now, we show that $f_1, f_2, \ldots, f_n$ a basis of V'.

→ To see that they are independent suppose that

$$\sum_{j=1}^{n} r_j f_j = 0$$

where $r_1, r_2, \ldots, r_n \in F$

$$\therefore \sum_{j=1}^{n} r_j f_j(v_i) = 0$$

$$\therefore \sum_{j=1}^{n} r_j (1) = 0$$

$$\therefore r_i = 0 \quad (\because j = i,\ f_i(v_i) = 1)$$

Thus $r_1, r_2, \ldots, r_n = 0$

and so $f_1, f_2, \ldots, f_n$ are LI.

To see that they span V'
let $f \in V'$

$f : V \to \mathbb{R}$

Let $v \in V$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

$\therefore f(v) = f(\alpha_1 v_1 + \cdots + \alpha_n v_n)$

$f(v) = f(\alpha_1 v_1) + f(\alpha_2 v_2) + \cdots + f(\alpha_n v_n)$

$f(v) = \alpha_1 f(v_1) + \alpha_2 f(v_2) + \cdots + \alpha_n f(v_n)$

$f(v) = f_1(v) f(v_1) + f_2(v) f(v_2) + \cdots + f_n(v) f(v_n)$

$(\because f_i(v) = \alpha_i)$

$$f(v) = \sum_{j=1}^{n} f_j(v) f(v_j)$$

$\therefore$ $f_1, f_2, \ldots, f_n$ is a basis of V' as the theorem states.

* Inner product *

Inner product :

For a given real vectorspace V, a mapping $f: V \times V \to \mathbb{R}$ is called an inner product if it satisfies the following properties for $x, y, z \in V$ and scalars $\alpha, \beta \in \mathbb{R}$

(i) $f(x,x) \geq 0$ & $f(x,x) = 0$ iff $x = 0$

(ii) $f(x,y) = f(y,x)$

(iii) $f(\alpha x + \beta y, z) = \alpha f(x,z) + \beta f(y,z)$

The real number $f(x,y)$ is called the inner product of vectors x and y.

Inner product space :

A real vectorspace V together with an inner product defined on it is called inner product space.

* In notation we will write $f(x,y)$ as $\langle x, y \rangle$ and $f: V \times V \to \mathbb{R}$ as $\langle , \rangle : V \times V \to \mathbb{R}$

* Examples *

Q-1

(1) Examine whether $\mathbb{R}^2$ will become inner product or not for the mapping $\langle , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as for $x = (a_1, a_2)$; $y = (b_1, b_2) \in \mathbb{R}^2$

(i) $\langle x, y \rangle = a_2 b_2$

(i) $f(x, x) \geq 0$ and $f(x, x) = 0$ iff $x = 0$

$\langle x, y \rangle = \langle (a_1, a_2), (b_1, b_2) \rangle$

$\langle x, x \rangle = \langle (a_1, a_2), (a_1, a_2) \rangle$

$\langle x, x \rangle = a_2^2 \geq 0$

$\Rightarrow a_2 = 0$

but we can take any value of a_1 for example $(2, 0)$ the $\langle x, x \rangle = 0$ and $(2, 0) \neq (0, 0)$ therefore first condition does not satisfies.

so, these product is not an inner product.

AIM :

(ii) $\langle x, y\rangle = a_1b_1 - 2a_2b_2$

$\langle x, y\rangle = \langle (a_1, a_2), (b_1, b_2)\rangle$

(i) $f(x, x) \geq 0$ & $f(x, x) = 0$ iff $x = 0$

$\langle x, x\rangle = \langle (a_1, a_2), (a_1, a_2)\rangle$

$\langle x, x\rangle = a_1^2 - 2a_2^2$

Let $x = (0, 1)$

$\langle (0,1), (0,1)\rangle = 0 - 2(1) = -2 < 0$

This product is not an inner product.

(iii) $\langle x, y\rangle = a_1b_1 + 2a_2b_2$

(i) $f(x, x) \geq 0$ & $f(x, x) = 0$ iff $x = 0$

$\langle x, x\rangle = \langle (a_1, a_2), (a_1, a_2)\rangle$

$\langle x, x\rangle = a_1^2 + 2a_2^2 \geq 0$

($\Rightarrow$) if $\langle x, x\rangle = 0$ then $x = 0$

$\langle x, x\rangle = a_1^2 + 2a_2^2 = 0$

AIM :

$$a_1^2 + 2a_2^2 = 0$$

$\therefore \quad a_1 = 0 \quad \& \quad a_2 = 0$

$\therefore \quad x = (a_1, a_2) = (0, 0) = \theta$

$(\Leftarrow)$ Let $x = \theta$ then $\langle x, x \rangle = 0$

$x = (a_1, a_2) = (0, 0)$

$\langle x, x \rangle = \langle (0,0), (0,0) \rangle$

$\langle x, x \rangle = 0 + 2(0)^2$

$\langle x, x \rangle = 0$

1^{st} $cond^n$ satisfies.

(ii) $\langle x, y \rangle = \langle y, x \rangle$

$\langle x, y \rangle = \langle (a_1, a_2), (b_1, b_2) \rangle$

$\langle x, y \rangle = a_1 b_1 + 2a_2 b_2$

$\langle x, y \rangle = b_1 a_1 + 2b_2 a_2$

$\langle x, y \rangle = \langle (b_1, b_2), (a_1, a_2) \rangle$

AIM :

$\langle x, y \rangle = \langle y, x \rangle$

$\therefore$ 2nd condn satisfies.

(iii) $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z)$

$f(\alpha x + \beta y, z)$

$x = (a_1, a_2)$, $y = (b_1, b_2)$, $z = (c_1, c_2)$

$\langle \alpha x + \beta y, z \rangle = \langle \alpha(a_1, a_2) + \beta(b_1, b_2), (c_1, c_2) \rangle$

$= \langle (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2), (c_1, c_2) \rangle$

$= (\alpha a_1 + \beta b_1) c_1 + 2(\alpha a_2 + \beta b_2) c_2$

$= \alpha a_1 c_1 + \beta b_1 c_1 + 2\alpha a_2 c_2 + 2\beta b_2 c_2$

$= \alpha a_1 c_1 + 2\alpha a_2 c_2 + \beta b_1 c_1 + 2\beta b_2 c_2$

$= \alpha(a_1 c_1 + 2a_2 c_2) + \beta(b_1 c_1 + 2b_2 c_2)$

$= \alpha \langle (a_1, a_2), (c_1, c_2) \rangle$
$+ \beta \langle (b_1, b_2), (c_1, c_2) \rangle$

$= \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

$\therefore$

$=$ RHS

Teacher's Signature :

Expt. No. ______ Page No. 6

Date ______

AIM :

3rd condⁿ satisfies.

This product is an inner product $\mathbb{R}^2$ is an inner product space.

Q - 2

(2) In the vectorspace $\mathbb{R}^2$, Let us define $\langle x, y \rangle$ as $\langle x, y \rangle = x_1y_1 + x_2y_2$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$ show that above product is inner product.

Solⁿ :- (i) $f(x, x) \geq 0$ & $f(x, x) = 0$ iff $x = \theta$

$\langle x, y \rangle = \langle (x_1, x_2), (y_1, y_2) \rangle$

$= x_1y_1 + x_2y_2$

$\langle x, x \rangle = \langle (x_1, x_2), (x_1, x_2) \rangle$

$\langle x, x \rangle = x_1^2 + x_2^2 \geq 0$

($\Rightarrow$) Let $\langle x, x \rangle = 0$ then $x = \theta$

$\langle x, x \rangle = x_1^2 + x_2^2 = 0$

$\therefore x_1 = 0, \; x_2 = 0$

Teacher's Signature :

Expt. No. ______ Date ______

AIM :

$\therefore x = (x_1, x_2) = (0, 0) = \theta$

$(\Leftarrow)$ if $x = \theta$ then $\langle x, x \rangle = 0$

$x = (x_1, x_2) = (0, 0)$

$\langle x, x \rangle = \langle (0,0), (0,0) \rangle$

$\langle x, x \rangle = 0 + 0$

$\langle x, x \rangle = 0$

1st cond^n satisfies.

(ii) $f(x, y) = f(y, x)$

$\langle x, y \rangle = x_1 y_1 + x_2 y_2$

$\langle x, y \rangle = y_1 x_1 + y_2 x_2$

$\langle x, y \rangle = \langle (y_1, y_2), (x_1, x_2) \rangle$

$\langle x, y \rangle = \langle y, x \rangle$

2nd cond^n satisfies.

AIM :

(iii) $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z)$

$x = (x_1, x_2)$; $y = (y_1, y_2)$; $z = (z_1, z_2)$

LHS $= \langle \alpha x + \beta y, z \rangle$

$= \langle \alpha(x_1, x_2) + \beta(y_1, y_2), (z_1, z_2) \rangle$

$= \langle (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2), (z_1, z_2) \rangle$

$= (\alpha x_1 + \beta y_1) z_1 + (\alpha x_2 + \beta y_2) z_2$

$= \alpha x_1 z_1 + \beta y_1 z_1 + \alpha x_2 z_2 + \beta y_2 z_2$

$= \alpha x_1 z_1 + \alpha x_2 z_2 + \beta y_1 z_1 + \beta y_2 z_2$

$= \alpha(x_1, z_1) + \alpha(x_2 z_2) + \beta y_1 z_1 + \beta y_2 z_2$

$= \alpha(x_1 z_1 + x_2 z_2) + \beta(y_1 z_1 + y_2 z_2)$

$= \alpha \langle (x_1, x_2), (z_1, z_2) \rangle + \beta \langle (y_1, y_2), (z_1, z_2) \rangle$

$= \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

$=$ RHS $\quad$ 3rd condn satisfies.

Dot product is an inner product

R^2 is an inner product space.

Teacher's Signature :

Q - 5

(3) consider the real vectorspace P_n (set of all polynomials of degree n). $(P_n = \{a_0 + a_1x_1 + \dots + a_nx_n\}$ $a_i \in \mathbb{R}\}$ and define $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx$ P_n is an inner product space or not.

Sol^n:- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = 0$

$$\langle p(x), p(x)\rangle = \int_0^1 p(x)\,p(x)\,dx$$

$$= \int_0^1 (p(x))^2\,dx \geq 0$$

$(\Rightarrow)$ since $(p(x))^2 \geq 0$

And if $p(x_0) \neq 0$, $x_0 \in [0,1]$

$p(x) \neq 0$ for $(x-\delta, x+\delta)$ & since polynomials are continuous.

So, $$\int_{x-\delta}^{x+\delta} (p(x))^2\,dx \neq 0$$ $\times$

$\therefore$ $p(x) = 0$ $\forall x \in [0,1]$

($\Leftarrow$) Take $p(x) = 0$

$\Rightarrow (p(x))^2 = 0$

$$\int_0^1 (p(x))^2 \, dx$$

$$= \int_0^1 (0) \, dx$$

$$= 0$$

1st condn satisfies.

(ii) $\langle p(x), q(x) \rangle = \int_0^1 p(x) q(x) \, dx$

$\langle p(x), q(x) \rangle = \int_0^1 q(x) p(x) \, dx$

$\langle p(x), q(x) \rangle = \langle (q(x)), p(x) \rangle$

2nd condn satisfies.

(iii) $\langle \alpha p(x) + \beta q(x), r(x) \rangle$

$= \alpha \langle p(x), r(x) \rangle + \beta \langle (q(x)), r(x) \rangle$

LHS : $\langle \alpha p(x) + \beta q(x), r(x) \rangle$

$$= \int_0^1 \alpha p(x) r(x)\, dx + \int_0^1 \beta q(x) r(x)\, dx$$

$$= \alpha \int_0^1 p(x) r(x)\, dx + \beta \int_0^1 q(x) r(x)\, dx$$

$$= \alpha \langle p(x), r(x) \rangle + \beta \langle q(x), r(x) \rangle$$

$=$ RHS

3^{rd} cond^n satisfies.

This product is an inner product
$P(n)$ is an inner product space.

* DEFINATIONS *

Norm of vector x :-

Norm of x is denoted by $\|x\|$ and defined as $\|x\| = \sqrt{\langle x, x \rangle}$

AIM :

(2) Length of vector x :-

Length of vector x is defined as $\|x - 0\| = \|x\|$

(3) Distance between two vectors x & y :-

Distance between two vectors x & y is denoted by $\|x-y\|$ and defined as $\|x-y\| = \sqrt{\langle x-y, x-y \rangle}$

* Unit vector :-

If $\|x\| = 1$ then x is called unit vector.

* Dot product is usual inner product for $\mathbb{R}^n$

* $\langle P(x), q(x) \rangle = \int_0^1 P(x)q(x)\,dx$ is usual inner product for P_n

* $\langle A, A \rangle = \text{Trace}(AA^T)$ is usual inner product for M_n.

AIM :

Q - 4

$(a_1, a_2) = a_1 b_1 + a_2 b_2$

(4) Inner product space = $\mathbb{R}^2$ with usual inner product find length of $(-1, 3)$

Sol^n:- Length of $x = \|x\| = \|(-1, 3)\|$

$$\|(-1,3)\| = \sqrt{\langle(-1,3),(-1,3)\rangle}$$

$$\|(-1,3)\| = \sqrt{(-1)^2 + (3)^2}$$

$$\|(-1,3)\| = \sqrt{1+9}$$

$$\|(-1,3)\| = \sqrt{10}$$

Length of $\|(-1,3)\|$ is $\sqrt{10}$

ઓળખાણ ~~પદ્ધતિ~~ ~~નકામી~~
કેમ ને પદ બનાવ
કોઈ એકનો વગ

Q - 5

(5) Inner product space = P_2 with usual inner product find length of $P(t) = 2 + 3t$

Sol^n:- Length of $P(t) = \|P(t)\|$

$$\|P(t)\| = \sqrt{\langle(P(t)), P(t)\rangle}$$

Expt. No. ______

AIM :

$$\|2+3t\| = \sqrt{\langle 2+3t, 2+3t\rangle}$$

$$= \sqrt{\int_0^1 (2+3t)^2 dt}$$

$$= \sqrt{\int_0^1 (4+12t+9t^2) dt}$$

$$= \sqrt{\left[4t + \frac{12t^2}{2} + \frac{9t^3}{3}\right]_0^1}$$

$$= \sqrt{4+6+3}$$

$$\|2+3t\| = \sqrt{13}$$

Length of $2+3t$ is $\sqrt{13}$

Q-6

ooo

(6) Show that vectorspace M2 (set of all 2×2 real matrices) define product as follows $\langle A, B\rangle$ = Trace (AB^T) = Trace (AB') for $A, B \in M2$ is an Inner product space.

AIM :

Sol^n: (i) $\langle A, A \rangle \geq 0$ & $\langle A, A \rangle = 0$ iff $A = \theta$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\langle A, A \rangle = Tr(AA^T)$$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$AA^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$AA^T = \begin{bmatrix} a^2+b^2 & ac+bd \\ ca+db & c^2+d^2 \end{bmatrix}$$

$$Tr(AA^T) = a^2+b^2+c^2+d^2 \geq 0$$

($\Rightarrow$) $\langle A, A \rangle = 0$ then $A = \theta$

$$\langle A, A \rangle = a^2+b^2+c^2+d^2 = 0$$

$$\therefore a=0, b=0, c=0, d=0$$

$$\therefore A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \theta$$

Expt. No. Date

AIM :

($\Leftarrow$) if $A = 0$ then $\langle A, A \rangle = 0$.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\langle A, A \rangle = Tr(AA^T)$$

$$AA^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Tr(AA^T) = 0$$

$$\therefore \langle A, A \rangle = 0$$

1^{st} cond^n satisfies.

(ii) $\langle A, B \rangle = \langle B, A \rangle$

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_1 & c_1 \\ b_1 & d_1 \end{bmatrix}, \quad B^T = \begin{bmatrix} a_2 & c_2 \\ b_2 & d_2 \end{bmatrix}$$

AIM :

$\langle A, B \rangle = Tr(AB^T)$

$$AB^T = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & c_2 \\ b_2 & d_2 \end{bmatrix}$$

$$AB^T = \begin{bmatrix} a_1a_2 + b_1b_2 & a_1c_2 + b_1d_2 \\ c_1a_2 + d_1b_2 & c_1c_2 + d_1d_2 \end{bmatrix}$$

$$Tr(AB^T) = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 \quad -(1)$$

$\langle B, A \rangle = Tr(BA^T)$

$$BA^T = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & c_1 \\ b_1 & d_1 \end{bmatrix}$$

$$BA^T = \begin{bmatrix} a_2a_1 + b_2b_1 & a_2c_1 + b_2d_1 \\ c_2a_1 + d_2b_1 & c_2c_1 + d_2d_1 \end{bmatrix}$$

$$Tr(BA^T) = a_2a_1 + b_2b_1 + c_2c_1 + d_2d_1 \quad -(2)$$

from (1) & (2)

$$\langle A, B \rangle = \langle B, A \rangle$$

(iii) $\langle \alpha A + \beta B, C \rangle = \alpha \langle A, C \rangle + \beta \langle B, C \rangle$

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}$$

$$C^T = \begin{bmatrix} a_3 & c_3 \\ b_3 & d_3 \end{bmatrix}$$

LHS : $\langle \alpha A + \beta B, C \rangle$

$$= \langle \alpha A + \beta B, C \rangle = Tr(\alpha A + \beta B)C^T$$

$$\alpha A + \beta B = \alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \beta \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$\alpha A + \beta B = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix} + \begin{bmatrix} \beta a_2 & \beta b_2 \\ \beta c_2 & \beta d_2 \end{bmatrix}$$

$$\alpha A + \beta B = \begin{bmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & \alpha d_1 + \beta d_2 \end{bmatrix}$$

$$(\alpha A + \beta B)C^T = \begin{bmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & \alpha d_1 + \beta d_2 \end{bmatrix} \begin{bmatrix} a_3 & c_3 \\ b_3 & d_3 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_1 a_3 + \beta a_2 a_3 + \alpha b_1 b_3 + \beta b_2 b_3 & \alpha a_1 c_3 + \beta a_2 c_3 + \alpha b_1 d_3 + \beta b_2 d_3 \\ \alpha c_1 a_3 + \beta c_2 a_3 + \alpha d_1 b_3 + \beta d_2 b_3 & \alpha c_1 c_3 + \beta c_2 c_3 + \alpha d_1 d_3 + \beta d_2 d_3 \end{bmatrix}$$

$$Tr(\alpha A + \beta B)C^T = \alpha a_1 a_3 + \beta a_2 a_3 + \alpha b_1 b_3 + \beta b_2 b_3 + \alpha c_1 c_3 + \beta c_2 c_3 + \alpha d_1 d_3 + \beta d_2 d_3 \quad (3)$$

Expt. No. ________

AIM :

RHS $= \alpha \langle A, C \rangle + \beta \langle B, C \rangle$

$= \alpha \, Tr(AC^T) + \beta \, Tr(BC^T)$

$$AC^T = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_3 & c_3 \\ b_3 & d_3 \end{bmatrix}$$

$$AC^T = \begin{bmatrix} a_1a_3 + b_1b_3 & a_1c_3 + b_1d_3 \\ c_1a_3 + d_1b_3 & c_1c_3 + d_1d_3 \end{bmatrix}$$

$$\alpha \, Tr(AC^T) = \alpha a_1a_3 + \alpha b_1b_3 + \alpha c_1c_3 + \alpha d_1d_3 \quad \text{—(3)}$$

$$BC^T = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_3 & c_3 \\ b_3 & d_3 \end{bmatrix}$$

$$BC^T = \begin{bmatrix} a_2a_3 + b_2b_3 & a_2c_3 + b_2d_3 \\ c_2a_3 + d_2b_3 & c_2c_3 + d_2d_3 \end{bmatrix}$$

$$\beta \, Tr(BC^T) = \beta a_2a_3 + \beta b_2b_3 + \beta c_2c_3 + \beta d_2d_3 \quad \text{—(4)}$$

Now, $\alpha \langle A, C \rangle + \beta \langle B, C \rangle$

$$= \alpha a_1a_3 + \alpha b_1b_3 + \alpha c_1c_3 + \alpha d_1d_3 + \beta a_2a_3 + \beta b_2b_3 + \beta c_2c_3 + \beta d_2d_3 \quad \text{—(5)}$$

Expt. No. ____ Page No. ____ Date ____

AIM :

From (3) & (5)

LHS = RHS

$\therefore$ 3^{rd} condⁿ satisfies.

This product is inner product.
M_e is an inner product space.

* Elementary properties of inner product *

Thm - 1

For vectors x, y, z in an inner product space V and scalars α, β:

(1) $\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$

Proof :- $\langle x, \alpha y + \beta z \rangle = \overline{\langle \alpha y + \beta z, x \rangle}$

($\because$ 2^{nd} prop. of IPS)

$\langle x, \alpha y + \beta z \rangle = \overline{\alpha \langle y, x \rangle + \beta \langle z, x \rangle}$

($\because$ 3^{rd} prop. of IPS)

Teacher's Signature : ____

$\langle x, \alpha y + \beta z \rangle = \alpha \langle x, z \rangle + \beta \langle x, z \rangle$

($\because$ 2nd prop. of IPS)

(2) $\langle \theta, x \rangle = \langle x, \theta \rangle = 0$

Proof :-

Let $x - x = \theta$

$\therefore \langle \theta, x \rangle = \langle x - x, x \rangle$

$\langle \theta, x \rangle = \langle x, x \rangle - \langle x, x \rangle$

$\langle \theta, x \rangle = 0$

This means inner product of any vectors with θ is 0.

(3) $\langle x, y \rangle = \langle x, z \rangle \Rightarrow y = z$

$L_{(1)}$

Proof :-

Let us take any $x \in V$

$\langle x, y - z \rangle = \langle x, y \rangle - \langle x, z \rangle$

$= 0$ (from (1))

Expt. No.
Page No.
Date
AIM :

$\langle x, 0 \rangle = 0$

$\therefore \quad y - z = 0$

$\therefore \quad y = z$

(iv) $\|\alpha x\| = |\alpha| \, \|x\|$

$\|\alpha x\|^2 = \langle \alpha x, \alpha x \rangle$

$\|\alpha x\|^2 = \alpha \langle x, \alpha x \rangle$

$\|\alpha x\|^2 = \alpha \langle \alpha x, x \rangle$

($\because$ 2nd prop of IPS)

$\|\alpha x\|^2 = \alpha^2 \langle x, x \rangle$

$\|\alpha x\| = |\alpha| \sqrt{\langle x, x \rangle}$

$\|\alpha x\| = |\alpha| \, \|x\|$

AIM :

(V) If $x \neq \theta$ and $\alpha = \frac{1}{\|x\|}$ then

$\|\alpha x\| = 1$ or αx is unit vector

for ex.

$$x = (1, 2)$$

$$\alpha x = \frac{x}{\|x\|}$$ ← multiplying both side with 'x'

$$\alpha x = \frac{(1, 2)}{\|(1, 2)\|}$$

$$\alpha x = \frac{(1, 2)}{\sqrt{1+4}}$$

$$\alpha x = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\|\alpha x\| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2}$$

$$\|\alpha x\| = \sqrt{\frac{1}{5} + \frac{4}{5}}$$

$$\|\alpha x\| = 1$$

Teacher's Signature :

Theorem - 2

For vectors $x_1, x_2, \ldots, x_n$; $y_1, y_2, \ldots, y_m \in U$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$; $\beta_1, \beta_2, \ldots, \beta_m$;

$$\left\langle \sum_{i=1}^{n} \alpha_i x_i , \sum_{j=1}^{m} \beta_j y_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \langle x_i, y_j \rangle$$

Theorem - 3

* For $x, y \in U$ and α, β are scalars.

(i) $\|\alpha x + \beta y\|^2 = \alpha^2 \|x\|^2 + 2\alpha\beta \langle x, y\rangle + \beta^2 \|y\|^2$

(ii) $4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2$

(Polarization identity)

Proof: (i) $\|\alpha x + \beta y\|^2 = \alpha^2 \|x\|^2 + 2\alpha\beta \langle x, y\rangle + \beta^2 \|y\|^2$

LHS $= \|\alpha x + \beta y\|^2$

$= \langle \alpha x + \beta y, \alpha x + \beta y \rangle$

$(\because \|x\|^2 = \langle x, x\rangle)$

Teacher's Signature :

$$= \alpha\langle x, \alpha x+\beta y\rangle + \beta\langle y, \alpha x+\beta y\rangle$$

(∵ 3rd prop. of IPS)

$$= \alpha\langle \alpha x+\beta y, x\rangle + \beta\langle \alpha x+\beta y, y\rangle$$

(∵ 2nd prop. of IPS)

$$= \alpha^2\langle x, x\rangle + \alpha\beta\langle y, x\rangle + \beta\alpha\langle x, y\rangle + \beta^2\langle y, y\rangle$$

(∵ 3rd prop. of IPS)

$$= \alpha^2\langle x, x\rangle + 2\alpha\beta\langle x, y\rangle + \beta^2\langle y, y\rangle$$

$$= \alpha^2\|x\|^2 + 2\alpha\beta\langle x, y\rangle + \beta^2\|y\|^2$$

$= \text{RHS}$

Hence proved.

(ii) $4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2$

$\text{LHS} = 4\langle x, y\rangle$

$\text{RHS} = \|x+y\|^2 - \|x-y\|^2$

AIM :

$= \langle x+y, x+y \rangle - \langle x-y, x-y \rangle$

$= \langle x, x+y \rangle + \langle y, x+y \rangle$
$- \langle x, x-y \rangle + \langle y, x-y \rangle$

$= \langle x+y, x \rangle + \langle x+y, y \rangle - \langle x-y, x \rangle$
$+ \langle y, x-y \rangle$

$= \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle$
$- \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle - \langle y, y \rangle$

$= 2\langle x, y \rangle + 2\langle y, x \rangle$

$= 4\langle x, y \rangle$ ($\because$ 2^{nd} prop of IPS)

$=$ LHS

Hence proved.

Expt. No. ____________ Date ____________

AIM :

Theorem - 4

* Cauchy schwartz inequality (Between inner product & norm):

For $x, y \in V$; V is an inner product space then

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Proof :-

Case - I

Let $\|y\| = 0$ — (i)

$\Rightarrow$ $y = \theta$

$\langle x, y \rangle = \langle x, \theta \rangle = 0$

$|\langle x, y \rangle| = 0 = \|x\| \|y\|$

$|\langle x, y \rangle| \leq \|x\| \|y\|$

Hence proved.

Teacher's Signature : ____________

Case - 2

Let $\|y\| \neq 0$

$$\|x - \alpha y\|^2 = \langle x - \alpha y, x - \alpha y \rangle$$

$$= \langle x, x - \alpha y \rangle - \alpha \langle y, x - \alpha y \rangle$$

($\because$ 3rd prop. of IPS)

$$= \langle x - \alpha y, x \rangle - \alpha \langle x - \alpha y, y \rangle$$

($\because$ 2nd prop. of IPS)

$$= \langle x, x \rangle - \alpha \langle y, x \rangle - \alpha \langle x, y \rangle + \alpha^2 \langle y, y \rangle$$

$$= \|x\|^2 - 2\alpha \langle x, y \rangle + \alpha^2 \|y\|^2 \quad (1)$$

Let $\alpha = \dfrac{\langle x, y \rangle}{\|y\|^2}$

from (1)

$$\|x - \alpha y\|^2 = \|x\|^2 - 2 \frac{\langle x, y \rangle}{\|y\|^2} \langle x, y \rangle + \frac{\langle x, y \rangle^2}{\|y\|^4} \|y\|^2$$

AIM :

$$\|x-\alpha y\|^2 = \|x\|^2 - 2\frac{\langle x, y\rangle^2}{\|y\|^2} + \frac{\langle x, y\rangle^2}{\|y\|^2}$$

$$\|x-\alpha y\|^2 = \|x\|^2 - \frac{\langle x, y\rangle^2}{\|y\|^2}$$

Since, $\|x-\alpha y\|^2 \geq 0$

$$\Rightarrow \|x\|^2 - \frac{\langle x, y\rangle^2}{\|y\|^2} \geq 0$$

$$\Rightarrow \|x\|^2 \geq \frac{\langle x, y\rangle^2}{\|y\|^2}$$

$$\Rightarrow \|x\|^2 \|y\|^2 \geq \langle x, y\rangle^2$$

$$\Rightarrow \langle x, y\rangle^2 \leq \|x\|^2 \|y\|^2$$

$$\Rightarrow |\langle x, y\rangle| \leq \|x\| \|y\|$$

Hence proved ...

VISION

Teacher's Signature :

Theorem - 5

* If x and y are linearly dependent vectors in U then $|\langle x, y\rangle| = \|x\|\|y\|$

Proof :-

Since x & y are dependent then $x = \alpha y$ for some α

$\therefore \quad x - \alpha y = \theta \quad \Rightarrow \|x - \alpha y\| = 0$

$\therefore \quad \|x - \alpha y\|^2 = 0$

$$\therefore \langle x-\alpha y, x-\alpha y\rangle = \langle x, x-\alpha y\rangle - \alpha\langle y, x-\alpha y\rangle$$

$$= \langle x-\alpha y, x\rangle - \alpha\langle x-\alpha y, y\rangle$$

$$= \langle x, x\rangle - \alpha\langle y, x\rangle - \alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle$$

$$= \langle x, x\rangle - 2\alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle$$

$$= \|x\|^2 - 2\alpha\langle x, y\rangle + \alpha^2\|y\|^2$$

Let $\alpha = \dfrac{\langle x, y\rangle}{\|y\|^2}$

$$\|x-\alpha y\|^2 = \|x\|^2 - 2\frac{\langle x,y\rangle}{\|y\|^2}\langle x,y\rangle + \frac{\langle x,y\rangle^2}{\|y\|^4}\|y\|^2$$

$$= \|x\|^2 - 2\frac{\langle x,y\rangle^2}{\|y\|^2} + \frac{\langle x,y\rangle^2}{\|y\|^2}$$

$$= \|x\|^2 - \frac{\langle x,y\rangle^2}{\|y\|^2}$$

Now, $\|x-\alpha y\|^2 = 0$

$$\Rightarrow \|x\|^2 - \frac{\langle x,y\rangle^2}{\|y\|^2} = 0$$

$$\Rightarrow \|x\|^2 = \frac{\langle x,y\rangle^2}{\|y\|^2}$$

$$\Rightarrow \|x\|^2\|y\|^2 = \langle x,y\rangle^2$$

$$\Rightarrow |\langle x,y\rangle| = \|x\|\|y\|$$

Hence proved.

AIM :

Theorem - 6

* Triangular inequality.

For $x, y \in U$ an inner product space

$$\|x+y\| \leq \|x\| + \|y\|$$

Proof :-

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$\|x+y\|^2 = \langle x, x+y \rangle + \langle y, x+y \rangle$$

$$\|x+y\|^2 = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$\|x+y\|^2 = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle$$

$$\|x+y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

($\because$ by cachy-schwartz's inequality)

$$\|x+y\|^2 \leq (\|x\| + \|y\|)^2$$

$$\|x+y\| \leq \|x\| + \|y\|$$

Hence proved

Teacher's Signature :

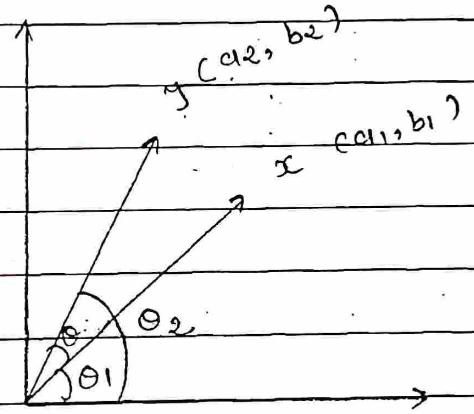
Expt. No.

AIM :

* Angle between two vectors x & y *

* Defination :-

Suppose two vectors $x, y \in U$
Angle between vectors is denoted by θ and it is defined as $\cos\theta = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}$



$$|\langle x, y\rangle| \leq \|x\|\,\|y\|$$

$$\Rightarrow \frac{|\langle x, y\rangle|}{\|x\|\,\|y\|} \leq 1$$

$$\Rightarrow -1 \leq \frac{\langle x, y\rangle}{\|x\|\,\|y\|} \leq 1$$

Let $\cos\theta = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}$

Expt. No. ______ Date ______

AIM :

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\|\|y\|}$$

$$\cos(\theta_2 - \theta_1) = \cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1$$

$$= \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}$$

$$= \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

Q - 7

(7) Find angle between $x = (1,0)$ & $y = (1,1)$ in $\mathbb{R}^2$.

Sol^n: Let us suppose that θ is an angle between x and y

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

$$\cos\theta = \frac{\langle (1,0), (1,1) \rangle}{\|(1,0)\| \|(1,1)\|}$$

Teacher's Signature : ______

Expt. No. ______ Date ______

AIM :

$$\cos\theta = \frac{(1,0)\cdot(1,1)}{\sqrt{(1)^2+(0)^2}\sqrt{(1)^2+(1)^2}}$$

$$\cos\theta = \frac{1+0}{\sqrt{1}\cdot\sqrt{2}}$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}$$

Angle between x & y is $\pi/4$

Q - 8

(8) Find angle between $x = (1,2,3)$, $y = (-3,0,1)$ in $\mathbb{R}^3$

Sol^n:- Let us suppose that θ is an angle between x & y

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

$$\cos\theta = \frac{\langle (1,2,3), (-3,0,1) \rangle}{\sqrt{1+4+9}\ \sqrt{9+0+1}}$$

$$\cos\theta = \frac{-3+0+3}{\sqrt{14}\ \sqrt{10}}$$

$$\cos\theta = \frac{0}{\sqrt{14}\sqrt{10}}$$

$$\cos\theta = 0$$

$$\theta = \cos^{-1} 0$$

$$\theta = \frac{\pi}{2}$$

$\therefore$ Angle between x & y is $\frac{\pi}{2}$

* Orthogonal vectors :-

Two vectors x & y are orthogonal to each other if $\langle x, y \rangle = 0$.

If a vector x is orthogonal to vector y in U then we denote it by $x \perp y$. Since $\langle x, y \rangle = \langle y, x \rangle$, $x \perp y$ implies $y \perp x$. Because of this we will say that vectors x & y are orthogonal to each other or simply orthogonal vectors.

Since $\langle \theta, x \rangle = \langle x, \theta \rangle$ for each $x \in U$, $\theta \perp x$. Moreover $x \perp x$ iff $x = \theta$

Theorem - 1

Pythagorean theorem :-

IMP

Statement :-

If $x \perp y$ in an inner product space U then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

In general, if vectors $x_1, x_2, \ldots, x_n$ are orthogonal to each other (i.e $x_i \perp x_j$ if $i \neq j$) then

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\|^2 = \sum_{j=1}^{n} |\alpha_j|^2 \|x_i\|^2$$

Proof :- $\left\| \sum_{i=1}^{n} \alpha_i x_i \right\|^2 = \left\langle \sum_{\substack{i=1 \\ j\neq i}}^{n} \alpha_i x_i , \sum_{i=1}^{n} \alpha_i x_i \right\rangle$

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\|^2 = \sum_{\substack{i=1 \\ j=1}}^{n} \alpha_i \alpha_j \langle x_i, x_j \rangle$$

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\|^2 = \sum_{i=1}^{n} \alpha_i^2 \langle x_i, x_i\rangle \quad \left[\langle x_i, x_j\rangle = 0 \text{ if } i \neq j\right]$$

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 \|x_i\|^2$$

* Definations :-

(1) **Orthogonal set :-**

A set $S = \{x_1, x_2, \dots, x_K\}$ is called an orthogonal set of vectors if $\langle x_i, x_j\rangle = 0$ for all $1 \leq i, j \leq K$ $i \neq j$

$= \|x\| \|y\| \cos\theta$

(2) **Orthonormal vectors :-**

Two vectors x_i & x_j are said to be orthonormal if $\langle x_i, x_j\rangle = 0$ and $\|x_i\| = \|x_j\| = 1$

Ex. $x = (0, 1)$, $y = (1, 0)$ are orthonormal vectors.

(3) **Orthonormal set :-**

A set $S = \{x_1, x_2, \dots, x_n\}$ is orthonormal if $\langle x_i, x_j\rangle = 0$ & $\|x_i\| = 1$ for all i, j, $1 \leq i, j \leq n$

(4) **Orthonormal basis :-**

IMP

A basis S of an inner product space U is called orthonormal basis of U

* if S is an orthonormal set in U.

ex. $\{x = (1,0),\ y = (0,1)\}$ is orthonormal basis for $\mathbb{R}^2$

$\mathbb{R}^3 = \{x = (1,0,0),\ y = (0,1,0),\ z = (0,0,1)\}$

Theorem - 2

Thm :- An orthonormal set S in an inner product space U is always linearly independent set.

IMP

Proof :- Let $S = \{x_1, x_2, ---, x_n\}$ be an orthonormal set in U.

Let $\alpha_1 x_1 + \alpha_2 x_2 + ---- + \alpha_n x_n = \theta$; --- (1)

$\langle \theta, x_i \rangle = 0$

$\therefore \langle \alpha_1 x_1 + \alpha_2 x_2 + --- + \alpha_n x_n , x_i \rangle$ (from (1))

$\therefore \alpha_1 \langle x_1, x_i \rangle + \alpha_2 \langle x_2, x_i \rangle + --- + \alpha_i \langle x_i, x_i \rangle + --- + \alpha_n \langle x_n, x_i \rangle$

$\langle x_i, x_i \rangle = 1$ $\therefore \alpha_i \langle x_i, x_i \rangle + \alpha_1 (0) + \alpha_2 (0) + --- + \alpha_n (0)$

$\langle x_i, x_j \rangle = 0$ $\therefore \alpha_i \langle x_i, x_i \rangle$

Pr

$\therefore \alpha_i (1)$

$\therefore \alpha_i = 0$

Similarly we can show that $\alpha_i = 0$ for $\forall i$

$\therefore$ S is linearly independent.

Theorem - 3

Thm Suppose U is a finite dimensional IPS with $\dim U = n$
An orthonormal subset S of U will be an orthonormal basis iff S contains n vectors.

Proof:- $\boxed{\begin{array}{l} ON \leftarrow S \subset U \rightarrow B = \{n \text{ elements}\} \\ \downarrow \qquad\quad \downarrow \\ \qquad\quad ONB \end{array}}$

ortho normal

$\Rightarrow$ S is ON (orthonormal) set & also ONB (orthonormal basis)
Since S is an basis of U and $\dim U = n$
$\Rightarrow$ S must have n vectors.

$\Leftarrow$ S is ON set with n vectors.
$\therefore$ By known result ON set must be a LI set.
(An orthonormal set S in IPS U is always LI set)
& Again by known result LI set wi n vectors must be a basis of U.

Imp

Theorem - 4 :-

Thm : IMP Suppose $S = \{x_1, x_2, \ldots, x_n\}$ is an ON set in IPS U. For $x \in U$.

(i) Vector $y = x - \sum_{i=1}^{n} \langle x, x_i \rangle x_i$ is orthogonal to each x_j ; $1 \leq j \leq n$

Proof :- Let $\langle y, x_j \rangle = \left\langle x - \sum_{i=1}^{n} \langle x, x_i \rangle x_i , x_j \right\rangle$;

$1 \leq j \leq n$

$$\langle y, x_j \rangle = \langle x, x_j \rangle - \left\langle \sum_{i=1}^{n} \langle x, x_i \rangle x_i , x_j \right\rangle$$

$$\langle y, x_i \rangle = \langle x, x_j \rangle - [\langle x, x_1 \rangle \langle x_1, x_j \rangle, \langle x, x_2 \rangle \langle x_2, x_j \rangle + \cdots + \langle x, x_j \rangle \langle x_j, x_j \rangle + \cdots + \langle x, x_n \rangle \langle x_n, x_j \rangle]$$

$$\langle y, x_i \rangle = \langle x, x_j \rangle - \langle x, x_j \rangle (1)$$

$$\langle y, x_i \rangle = \langle x, x_j \rangle - \langle x, x_j \rangle$$

$$\langle y, x_i \rangle = 0$$

(ii) If $x \notin [x_1, x_2, \ldots, x_r]$ then $y \neq \theta$ & taking $x_{r+1} = \frac{y}{\|y\|}$; the set

IMP

$\{x_1, x_2, \ldots, x_r, x_{r+1}\}$ is also an orthonormal set.

Proof :- Let us suppose that $y = \theta$

$$y = x - \sum_{i=1}^{r} \langle x, x_i \rangle x_i = \theta$$

$$\therefore \quad x = \sum_{i=1}^{r} \langle x, x_i \rangle x_i$$

$$\therefore \quad x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r$$

$$; \quad \alpha_i = \langle x, x_i \rangle$$

$$\therefore \quad x \in [x_1, x_2, \ldots, x_r] \quad ※$$

$$\therefore \quad y \neq \theta \Rightarrow \|y\| \neq 0$$

$$\Rightarrow \frac{y}{\|y\|} \text{ is a well defined number.}$$

Let $x_{r+1} = \frac{y}{\|y\|} \Rightarrow \|x_{r+1}\| = \frac{\|y\|}{\|y\|} = 1$

Let $\langle x_{r+1}, x_j \rangle = \left\langle \frac{x - \sum_{i=1} \langle x_j, x_i \rangle x_i}{\|y\|}, x_j \right\rangle$;

$j = 1, 2, \ldots, r$

$\therefore \langle x_{n+1}, x_j \rangle = \frac{1}{\|y\|} \langle x, x_j \rangle - \frac{1}{\|y\|} \sum_{i=1}^{n} \langle x, x_i \rangle \langle x_i, x_j \rangle$

$= \frac{\langle x, x_j \rangle}{\|y\|} - \frac{1}{\|y\|} \sum_{i=1}^{n} \langle x, x_i \rangle \langle x_i, x_j \rangle$

$= \frac{\langle x, x_j \rangle}{\|y\|} - \frac{1}{\|y\|} \{ \langle x, x_1 \rangle \langle x_1, x_j \rangle + \langle x, x_2 \rangle \langle x_2, x_j \rangle + \cdots + \langle x, x_j \rangle \langle x_j, x_j \rangle + \cdots + \langle x, x_n \rangle \langle x_n, x_j \rangle$

$= \frac{\langle x, x_j \rangle}{\|y\|} - \frac{\langle x, x_j \rangle (1)}{\|y\|}$

$= 0$

$\therefore \{x_1, x_2, \ldots, x_n, x_{n+1}\}$ is an orthonormal set.

Theorem - 5

Thm :- A finite dimensional IPS possesses an orthonormal basis.

Proof :- Consider a non zero vector $y_1 \in U$
Then consider $x_1 = \frac{y_1}{\|y_1\|} \Rightarrow$

$\{x_1\}$ is an orthonormal set.

$\Rightarrow$ If $\dim U = n = 1$ then $\{x_1\}$ is the required basis.

Suppose $\dim U \neq 1 \Rightarrow U \neq [x_1]$

Let $y_2 \in U - [x_1]$

Let $x_2 = \frac{y_2}{\|y_2\|} \Rightarrow \{x_1, x_2\}$ is an orthonormal set.

If $\dim U = n = 2$ then we are done & $\{x_1, x_2\}$ is a basis.

If this is not then continue the procedure untill we get the basis.

* Gram - schmidt orthonormalization process.

IMP Suppose $B = \{u_1, u_2, \ldots, u_n\}$ is a linear basis of an inner product space U. Then there exist a set $B_n = \{x_1, x_2, \ldots, x_n\}$ such that B_n is again basis for U & $[B_n] = [y_1, y_2, \ldots, y_n]$ where $y_i = \frac{x_i}{\|x_i\|}$, thus $\{y_1, y_2, \ldots, y_n\}$ is the required orthonormal basis for U.

Q-1

(1) Consider linear basis $B = \{(1,2),(5,-1)\}$ of euclidean space $\mathbb{R}^2$ and obtain its orthonormal basis.

Sol^n :- Let $B = \{(1,2),(5,-1)\}$

$$x_1 = v_1 = (1,2)$$

$$y_1 = \frac{x_1}{\|x_1\|}$$

$$y_1 = \frac{(1,2)}{\sqrt{1+4}}$$

$$y_1 = \frac{(1,2)}{\sqrt{5}}$$

$$y_1 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$x - \sum_{i=1}^{r} \langle x, y_i \rangle y_i$$

$$x_2 = \underline{v_2 - \langle v_2, y_1 \rangle y_1}$$

$$= (5,-1) - \langle (5,-1), (1/\sqrt{5}, 2/\sqrt{5}) \rangle \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$= (5,-1) - \frac{3}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$= (5,-1) - \left(\frac{3}{5}, \frac{\cancel{2}\,6}{5}\right)$$

$$x_2 = \left(\frac{22}{5}, \frac{-11}{5}\right)$$

$$\Rightarrow \quad y_2 = \frac{x_2}{\|x_2\|}$$

$$y_2 = \frac{\left(\frac{22}{5}, \frac{-11}{5}\right)}{\sqrt{\left(\frac{22}{5}\right)^2 + \left(\frac{-11}{5}\right)^2}}$$

$$y_2 = \frac{\left(\frac{22}{5}, \frac{-11}{5}\right)}{\frac{11}{5}\sqrt{(2)^2 + (1)^2}}$$

$$y_2 = \frac{\frac{11}{5}(2, -1)}{\frac{11}{5}\sqrt{5}}$$

$$y_2 = \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$$

$\therefore$ Required orthonormal basis : $\{y_1, y_2\}$

$$= \left\{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)\right\}$$

Q-2

Q-2 Consider the basis $B = \{v_1 = (0,1,1), v_2 = (1,0,1), v_3 = (1,1,0)\}$ of $\mathbb{R}^3$ obtain orthonormal basis by Gram schmidt process.

Soln :- Let $x_1 = v_1 = (0,1,1)$

$$y_1 = \frac{x_1}{\|x_1\|}$$

$$y_1 = \frac{(0,1,1)}{\sqrt{1+1}}$$

$$y_1 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$x_2 = v_2 - \langle v_2, y_1 \rangle y_1$$

$$x_2 = (1,0,1) - \left\langle (1,0,1), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$x_2 = (1,0,1) - \frac{1}{\sqrt{2}}\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$x_2 = (1,0,1) - \left(0, \frac{1}{2}, \frac{1}{2}\right)$$

$$x_2 = \left(1, \frac{-1}{2}, \frac{1}{2}\right)$$

$$y_2 = \frac{x_2}{\|x_2\|}$$

$$y_2 = \frac{\left(1, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{(1)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}}$$

$$y_2 = \frac{\left(1, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{1 + \frac{1}{4} + \frac{1}{4}}}$$

$$y_2 = \frac{\left(1, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{6}{4}}}$$

$$y_2 = \frac{\left(1, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{\frac{3}{2}}}$$

$$y_2 = \left(1, -\frac{1}{2}, \frac{1}{2}\right) \cdot \frac{\sqrt{2}}{\sqrt{3}}$$

$$y_2 = \left(\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$x_3 = v_3 - \langle v_3, y_1 \rangle y_1 - \langle v_3, y_2 \rangle y_2$$

$$x_3 = (1,1,0) - \left\langle (1,1,0), \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\rangle \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$- \left\langle (1,1,0), \left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \right\rangle \left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$x_3 = (1,1,0) - \frac{1}{\sqrt{2}}\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$- \frac{1}{\sqrt{6}}\left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$x_3 = (1,1,0) - \left(0, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{3}, \frac{-1}{6}, \frac{1}{6}\right)$$

$$x_3 = \left(\frac{2}{3}, \frac{2}{3}, \frac{-2}{3}\right)$$

$$y_3 = \frac{x_3}{\|x_3\|}$$

$$y_3 = \frac{\left(\frac{2}{3}, \frac{2}{3}, \frac{-2}{3}\right)}{\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{-2}{3}\right)^2}}$$

$$y_3 = \frac{2/3\,(1, 1, -1)}{\frac{2}{3}\sqrt{1+1+1}}$$

$$Y_3 = \frac{(1, 1, -1)}{\sqrt{3}}$$

$$Y_3 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right)$$

$\therefore$ Required orthonormal basis is :

$$\left\{ \left(1, \frac{-1}{2}, \frac{1}{2}\right), \left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \right\}$$

Q-3

3) Consider the linear basis $B = \{v_1 = 1, v_2 = 1+t, v_3 = 1+t+t^2\}$ of vectorspace P_2 with inner product $\langle p(t), q(t) \rangle = \int_0^1 p(t)q(t)\,dt$. Obtain orthonormal basis of P_2

Solⁿ :- $x_1 = v_1 = 1$

$$Y_1 = \frac{x_1}{\|x_1\|} = 1$$

$$x_2 = v_2 - \langle v_2, y_1 \rangle y_1$$

$$x_2 = 1+t - \langle 1+t, 1 \rangle 1$$

$$\langle 1+t, 1 \rangle = \int_0^1 (1+t)(1)\,dt$$

$$\langle 1+t, 1 \rangle = \left[t + \frac{t^2}{2} \right]_0^1$$

$$\langle 1+t, 1\rangle = 1 + \frac{1}{2} = \frac{3}{2}$$

$$x_2 = (1+t) - \frac{3}{2}$$

$$x_2 = t + 1 - \frac{3}{2}$$

$$x_2 = t - \frac{1}{2}$$

$$y_2 = \frac{x_2}{\|x_2\|}$$

$$y_2 = \frac{t - \frac{1}{2}}{\left\| t - \frac{1}{2} \right\|}$$

$$\left\| t - \frac{1}{2} \right\|^2 = \left\langle t - \frac{1}{2}, t - \frac{1}{2} \right\rangle$$

$$= \int_0^1 \left(t - \frac{1}{2}\right)^2 dt$$

$$= \int_0^1 \left(t^2 - t + \frac{1}{4}\right) dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{4}\right]_0^1$$

$$\left\| t - \frac{1}{2} \right\|^2 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

$$y_2 = \frac{t - \frac{1}{2}}{\sqrt{1/12}}$$

$$y_2 = \frac{t - \frac{1}{2}}{1/2\sqrt{3}}$$

$$y_2 = 2\sqrt{3}\left(t - \frac{1}{2}\right)$$

$$y_2 = 2\sqrt{3}t - \frac{2\sqrt{3}}{2}$$

$$y_2 = 2\sqrt{3}t - \sqrt{3}$$

$$x_3 = v_3 - \langle v_3, y_1 \rangle y_1 - \langle v_3, y_2 \rangle y_2$$

$$= 1 + t + t^2 - \langle 1 + t + t^2, 1 \rangle 1$$
$$- \langle 1 + t + t^2, 2\sqrt{3}t - \sqrt{3} \rangle$$
$$(2\sqrt{3}t - \sqrt{3})$$

$$\langle 1 + t + t^2, 1 \rangle = \int_0^1 (1 + t + t^2) 1 \, dt$$

$$\langle 1 + t + t^2, 1 \rangle = \left[t + \frac{t^2}{2} + \frac{t^3}{3} \right]_0^1$$

$$\langle 1 + t + t^2, 1 \rangle = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

$$\langle 1+t+t^2,\ 2\sqrt{3}t-\sqrt{3}\rangle = \int_0^1 (1+t+t^2)(2\sqrt{3}t-\sqrt{3})\,dt$$

$$= \int_0^1 \left(2\sqrt{3}t + 2\sqrt{3}t^2 + 2\sqrt{3}t^3 - \sqrt{3} - \sqrt{3}t - \sqrt{3}t^2\right) dt$$

$$= \left[\frac{2\sqrt{3}t^2}{2} + \frac{2\sqrt{3}t^3}{3} + \frac{2\sqrt{3}t^4}{4} - \sqrt{3}t - \frac{\sqrt{3}t^2}{2} - \frac{\sqrt{3}t^3}{3}\right]_0^1$$

$$= \sqrt{3} + \frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{2} - \sqrt{3} - \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

$$x_3 = 1+t+t^2 - \frac{11}{6} - \frac{1}{\sqrt{3}}(2\sqrt{3}t - \cancel{t}\sqrt{3})$$

$$x_3 = 1+t+t^2 - \frac{11}{6} - 2t + 1$$

$$x_3 = t^2 - t + 2 - \frac{11}{6} = t^2 - t + \frac{1}{6}$$

$$x_3 = t^2 - t + 1/6$$

4 $$y_3 = \frac{x_3}{\|x_3\|}$$

$$y_3 = \frac{t^2 - t + \frac{1}{6}}{\left\| t^2 - t + \frac{1}{6} \right\|}$$

$$\left\| t^2 - t + \frac{1}{6} \right\|^2 = \left\langle t^2 - t + 1/6 ,\ t^2 - t + 1/6 \right\rangle$$

$\frac{1}{\sqrt{3}}$

$$= \int_0^1 \left(t^2 - t + 1/6\right)\left(t^2 - t + 1/6\right) dt$$

$$= \int_0^1 (t^2 - t + 1/6)^2 \, dt$$

$$= \int_0^1 \left(t^4 + t^2 + \frac{1}{36} - 2t^3 - \frac{t}{3} + \frac{2t^2}{6} \right) dt$$

$$= \left[\frac{t^5}{5} + \frac{t^3}{3} + \frac{t}{36} - \frac{2t^4}{4} - \frac{t^2}{6} + \frac{t^3}{9} \right]_0^1$$

$$= \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{1}{2} - \frac{1}{6} + \frac{1}{9}$$

$$= \frac{1}{5} - \frac{7}{36}$$

$$= \frac{1}{180}$$

$$\|t^2 - t + \tfrac{1}{6}\| = \frac{1}{\sqrt{180}} = \frac{1}{3\sqrt{20}} = \frac{1}{6\sqrt{5}}$$

$$\therefore \quad Y_3 = \frac{(t^2 - t + 1/6)}{1/6\sqrt{5}}$$

$$Y_3 = 6\sqrt{5}\,(t^2 - t + 1/6)$$

$\therefore$ Riquesred orthonormal basis is

$$\{1,\ 2\sqrt{3}\,(t - 1/2),\ 6\sqrt{5}\,(t^2 - t + 1/6)\}$$

* Definations

(1) orthogonal vectorx to a set V:

Suppose V is a non empty subset of an inner product space U. A vector $x \in U$ is said to be orthogonal to V if $\langle x, y \rangle = 0$ for all $y \in V$.

(2) Orthogonal set to a set :

For given nonempty subset V & W of an inner product space U, we say that V is orthogonal to W if each vector of V is orthogonal to W.

* Orthogonal complement :-

IMP

For a given non empty subset V of an inner product space U, the set $\{x \in U \mid x \perp V\}$ is called an orthogonal complement of V & it is denoted by $V^{\perp}$ "V perp."

* Examples *

(1) $U = \mathbb{R}^2$; $V = \{(1,1)\}$. Find $V^{\perp}$

$V^{\perp} = (1,1)^{\perp}$

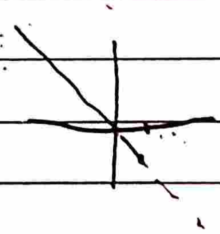
$(1,1)^{\perp} = \{(a,b) \in \mathbb{R}^2 \mid \langle (a,b), (1,1) \rangle = 0\}$

$\langle (a,b), (1,1) \rangle = 0$

$a+b = 0$

$a = -b$ & $b = -a$

$\therefore V^{\perp} = \{(a,-a) \mid a \in \mathbb{R}\}$



$x = -y$ line

(2) $\{\theta\}^{\perp}$; where $\theta \in U$

$\{\theta\}^{\perp} = \{x \in U / \langle x, \theta \rangle = 0\}$

$= U$.

(3) $\{(3,5)\}^{\perp}$ where $(3,5) \in \mathbb{R}^2$

Sol^n : $V = \{(3,5)\}$

$V^{\perp} = \{(3,5)\}^{\perp}$

$V^{\perp} = \{(a,b) \in \mathbb{R}^2 \mid \langle (a,b),(3,5) \rangle = 0\}$

$\langle (a,b),(3,5) \rangle = 0$

$3a + 5b = 0$

$5b = -3a$

$b = \frac{-3}{5} a$

$V^{\perp} = \{(a, -\tfrac{3}{5} a) \mid a \in \mathbb{R}\}$

$y = \frac{-3}{5} x$ line

(4) $\{(a,0,0) \mid a \in \mathbb{R}\} = V \subset \mathbb{R}^3 = 0$

Sol^n : $V^{\perp} = \{(a,b,c) \in \mathbb{R}^3 \mid \langle (a,b,c),(a,0,0) \rangle = 0\}$

$\langle (a,b,c),(a,0,0) \rangle = 0$

$\therefore a^2 = 0$

$\therefore a = 0$

$V^{\perp} = \{(a,b,c) \in \mathbb{R}^3 \mid a = 0\}$

Y-Z plane

(5) U = IPS then find $U^\perp$

$U^\perp = \{0\}$

(6) Let $\mathbb{R}^2$ with inner product defined as follows : $\langle x, y\rangle = \langle (a_1, a_2), (b_1, b_2)\rangle$
$= a_1b_1 + a_1b_2 + a_2b_1 + 2a_2b_2$
Find $V^\perp$ for $V = \{x = (a_1, a_2) \mid a_1 = a_2\}$

Sol^n :- $V^\perp = \{(b_1, b_2) \in \mathbb{R}^2 \mid \langle (a_1, a_2), (b_1, b_2)\rangle = 0\}$
$\langle (b_1, b_2), (a_1, a_2)\rangle = 0$
$\langle (a_1, a_2), (b_1, b_2)\rangle = 0$

$\therefore \quad a_1b_1 + a_1b_2 + a_2b_1 + 2a_2b_2 = 0$

but $a_1 = a_2$

$\therefore \quad a_1b_1 + a_1b_2 + a_1b_1 + 2a_1b_2 = 0$

$\therefore \quad a_1(2b_1 + 3b_2) = 0$

$\therefore \quad a_1 = 0 \quad$ or $\quad b_1 = \dfrac{-3b_2}{2}$
but $a_1 \neq a$

$b_2 = \dfrac{-2}{3}b_1$

$V^\perp = \{(b_1, -2/3\, b_1) \mid b_1 \in \mathbb{R}\}$

$y = -2/3\, x$ line

$V^\perp = \{(b_1, b_2) \mid 3b_2 + 2b_1 = 0\}$

Q - 2

(2) obtain $A^{\perp}$ in the IPS M_2 for $A = \begin{bmatrix} -1 & 2 \\ 5 & 3 \end{bmatrix}$

IMP

with the IPS defined as $\langle A, B \rangle = Tr(AB^T)$

Sol^n :- Let $A^{\perp} = \left\{ B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2 \;/\right.$

$$\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 5 & 3 \end{bmatrix} \right\rangle = 0$$

$$\therefore \left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 5 & 3 \end{bmatrix} \right\rangle = 0$$

$$\langle A, B \rangle = Tr(AB^T)$$

$$B^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$AB^T = \begin{bmatrix} -1 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$AB^T = \begin{bmatrix} -a + 2c & -b + 2d \\ 5a + 3c & 5b + 3d \end{bmatrix}$$

$$Tr(AB^T) = -a + 2c + 5b + 3d$$

Let $\langle A, B \rangle = 0$

$\therefore \quad -a + 2c + 5b + 3d = 0$

$\therefore \quad a = 2c + 5b + 3d$

Let $c = r, \quad b = s, \quad d = t$

$$A^{\perp} = \left\{ \begin{bmatrix} 2r + 5s + 3T & r \\ s & t \end{bmatrix} \middle| \; r, s, t \in \mathbb{R} \right\}$$

$$= \left\{ r \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + s \begin{bmatrix} 5 & 0 \\ 1 & 0 \end{bmatrix} + t \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \middle| \; r, s, t \in \mathbb{R} \right\}$$

Spanned by $\left\{ \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Theorem 6 :

Thm :- For subspaces V & W of IPS U

(i) $(V+W)^{\perp} = V^{\perp} \cap W^{\perp}$

(ii) $V^{\perp} + W^{\perp} = (V \cap W)^{\perp}$

Sol^n :- $[A = B, \; A \subset B, \; B \subset A]$

Proof :-

(i) Let $x \in (V+W)^{\perp}$

$\Rightarrow$ For every $y \in V+W$, $\langle x, y \rangle = 0$

$\Rightarrow$ $y = r + s$ for $r \in V$ & $s \in W$

Since $r \in V \subset V+W \Rightarrow r \in V+W$

$\Rightarrow \quad \langle x, r \rangle = 0$

$\Rightarrow \quad x \in V^{\perp}$

Similarly, $s \in V+W$ & $\langle x, s \rangle = 0$

$\Rightarrow \quad x \in W^{\perp}$

$\therefore \quad x \in (V^{\perp} \cap W^{\perp})$

$\Rightarrow$ Let $x \in V^{\perp} \cap W^{\perp}$

$x \in V^{\perp}$ & $x \in W^{\perp}$

$\Rightarrow \langle x, v \rangle = 0$ & $\langle x, w \rangle = 0$

for all $v \in V$ & $w \in W$

Let $y \in V+W$

$\langle x, y \rangle = \langle x, v+w \rangle$

$\langle x, y \rangle = \langle x, v \rangle + \langle x, w \rangle$

$\langle x, y \rangle = 0$

$\therefore \quad x \in (V+W)^{\perp}$

$\therefore \quad V^{\perp} \cap W^{\perp} \subset (V+W)^{\perp}$

$\therefore \quad V^{\perp} \cap W^{\perp} = (V+W)^{\perp}$

(ii) Since V & W are subspaces of U therefore $V^{\perp}$ & $W^{\perp}$ are also subspaces of U.

$\Rightarrow$ (i) result is true for $V^{\perp}$ & $W^{\perp}$ also (∵ $V^{\perp}$ & $W^{\perp}$ are subspaces.)

Let us replace V & W by $V^{\perp}$ & $W^{\perp}$ in (i) respectively.

$$(V^{\perp}+W^{\perp})^{\perp} = (V^{\perp})^{\perp} \cap (W^{\perp})^{\perp}$$

$$(V^{\perp}+W^{\perp})^{\perp} = V \cap W$$

Let us take $\perp$ both the sides.
we get,

$$V^{\perp} + W^{\perp} = (V \cap W)^{\perp}$$

* Orthogonal transformation :-

A linear transformation $T : U \to U$ is said to be orthogonal if $\|T(x)\| = \|x\|$ for $x \in U$

IMP

Theorem - 7

A linear transformation $T: U \to U$ is orthogonal iff T maps on an orthonormal basis of U (domain) to orthonormal basis of U (codomain).

Proof :- ($\Longrightarrow$) Let T be an orthogonal transformation — (1)

$\therefore$ By def^n $\|T(x)\| = \|x\|$ — (2)

Let $B = \{x_1, x_2, ---, x_n\}$ be an orthonormal basis for U (domain).

$\therefore$ $B' = \{T(x_1), T(x_2), ---, T(x_n)\}$ be a set in U (codomain)

$$\|T(x_i)\| = \|x_i\| = 1 \quad \text{— (3)}$$

$$\|T(x_i - x_j)\| = \|x_i - x_j\|$$

$$\Rightarrow \|T(x_i - x_j)\|^2 = \|x_i - x_j\|^2$$

$$\Rightarrow \|T(x_i) - T(x_j)\|^2 = \|x_i - x_j\|^2$$

$$(\because T(x+y) = T(x) + T(y))$$

$$\Rightarrow \langle T(x_i) - T(x_j), T(x_i) - T(x_j)\rangle = \langle x_i - x_j, x_i - x_j\rangle$$

$$\Rightarrow \langle T(x_i), T(x_i)\rangle - \langle T(x_j), T(x_i)\rangle - \langle T(x_i), T(x_j)\rangle + \langle T(x_j), T(x_j)\rangle$$

$$= \langle x_i, x_j \rangle - \langle x_i, x_j \rangle - \langle x_j, x_i \rangle + \langle x_j, x_j \rangle$$

$$\Rightarrow \quad \|T(x_i)\|^2 - 2\langle T(x_i), T(x_j) \rangle + \|T(x_j)\|^2$$

$$= \|x_i\|^2 - 2\langle x_i, x_j \rangle + \|x_j\|^2$$

$$\Rightarrow \quad \|x_i\|^2 - 2\langle T(x_i), T(x_j) \rangle + \|\cancel{T}(x_j)\|^2$$

$$= \|x_i\|^2 - 2\langle x_i, x_j \rangle + \|x_j\|^2$$

$$\Rightarrow \quad \langle T(x_i), T(x_j) \rangle = \langle x_i, x_j \rangle = 0$$

for $i \neq j$

(span કરે છે LI છે Hence basis)

$\therefore$ B' is the required orthonormal basis.

$(\Leftarrow)$ Let $x \in U$ (domain)

$$\Rightarrow \quad x = \sum_{i=1}^{n} \alpha_i x_i \quad \text{--- (4)}$$

Since $B = \{x_1, x_2, \ldots, x_n\}$ mapped to $B' = \{T(x_1), T(x_2), \ldots, T(x_n)\}$ basis for U (co-domain) by T.

For any $T(x) \in U$ (co-domain)

$$\therefore T(x) = \sum_{i=1}^{n} \alpha_i T(x_i)$$

$$\|T(x)\| = \left\| \sum_{i=1}^{n} \alpha_i T(x_i) \right\|$$

$$\|T(x)\| = \sum_{i=1}^{n} |\alpha_i| \, \|T(x_i)\|$$

$$\|T(x)\| = \sum_{i=1}^{n} |\alpha_i| \, \|x_i\|$$

$$(\because \|(T(x_i))\| = \|x_i\|)$$

$$\|T(x)\| = \|x\|$$

Theorem - 8

Imp

The following are equivalent for linear transformation $T : U \to U$

(i) T is orthogonal

(ii) $\|T(x) - T(y)\| = \|x - y\|$; $x, y \in U$

(iii) $\langle T(x), T(y) \rangle = \langle x, y \rangle$; $x, y \in U$

(iv) $T'T = TT' = I$

Solⁿ :- (i) → (ii) it is given that T is orthogonal

we will show that $\|T(x) - T(y)\| = \|x - y\|$

LHS : $\|T(x) - T(y)\|$

$= \| T(x) + (-T)(y) \|$

$= \| T(x) + T(-y) \|$

$= \| T(x-y) \|$ $(\because T \text{ is LT})$

$= \| x-y \|$ $(\because$ by def^n of orthogonal transformation$)$

$=$ RHS

(a) (ii) → (iii) it is given that $\| T(x) - T(y) \| = \| x-y \|$ we will show that $\langle T(x), T(y) \rangle = \langle x, y \rangle$

$$\| T(x) - T(y) \| = \| x-y \|$$

Squaring both the sides, we get

$$\| T(x) - T(y) \|^2 = \| x-y \|^2$$

$$\langle T(x) - T(y), T(x) - T(y) \rangle = \langle x-y, x-y \rangle$$

$$\therefore \langle T(x), T(x) \rangle - \langle T(x), T(y) \rangle - \langle T(y), T(x) \rangle + \langle T(y), T(y) \rangle$$

$$= \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle$$

$$\therefore \| T(x) \|^2 - 2\langle T(x), T(y) \rangle + \| T(y) \|^2 = \| x \|^2 - 2\langle x, y \rangle + \| y \|^2$$

$$\therefore \| x \|^2 - 2\langle T(x), T(y) \rangle + \| y \|^2 = \| x \|^2 - 2\langle x, y \rangle + \| y \|^2$$

$(\because T$ is O.T.$)$

$\therefore \quad -2\langle T(x), T(y)\rangle = -2\langle x, y\rangle$

$\therefore \quad \langle T(x), T(y)\rangle = \langle x, y\rangle$

(iii) $\rightarrow$ (iv)

It is given that $\langle x, y\rangle = \langle T(x), T(y)\rangle$

We will show that $T'T = TT' = I$

$$\langle x, y\rangle = \langle T(x), T(y)\rangle$$

$$\langle x, y\rangle = \langle x, T'T(y)\rangle \quad (\because \text{O.T})$$

$$\langle x, y\rangle = \langle x, (T'T)y\rangle$$

$$y = (T'T)y$$

$$\therefore \ y = Iy \qquad \therefore \ T'T = I$$

Now, we will show that the existance of T'.
First we prove that oneoneness

Let us take $\quad T(x) = T(y)$

$$\therefore \ T(x) - T(y) = \theta$$

$$\therefore \ \|T(x) - T(y)\| = 0$$

$$\therefore \ \|T(x-y)\| = 0$$

$$\therefore \ \|x-y\| = 0 \quad (\because \text{LT})$$

$$\therefore \ x - y = \theta$$

$$\therefore \ x = y$$

T is one one → T is non singular

$\therefore$ $T'T = TT' = I$

(iv) → (i)

it is given that $T'T = TT' = I$

We will show that T is orthogonal.

$\|T(x)\| = \|T(x)\|$

$$\|T(x)\|^2 = \langle T(x), T(x) \rangle$$

$$\|T(x)\|^2 = \langle x, T'T(x) \rangle$$

$$\|T(x)\|^2 = \langle x, (T'T) x \rangle$$

$$\|T(x)\|^2 = \langle x, I x \rangle$$

$$\|T(x)\|^2 = \langle x, x \rangle \quad (\because x = Ix)$$

$$\|T(x)\|^2 = \|x\|^2$$

$$\|T(x)\| = \|x\|$$

T is orthogonal LT