

* Eigen values & Eigen vectors

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→ Definition :-

Let A be a given square matrix say $n \times n$ then there exists scalar λ and non zero vector X . Such that $AX = \lambda X$ then λ is called eigen value of matrix A or latent value or characteristic value or proper value.

And $X \neq 0$ is called eigen vector

→ X is a column matrix.

$$AX = \lambda X$$

$$\Rightarrow AX - \lambda X = 0$$

$$\therefore X(A - \lambda I) = 0$$

$$[I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}]$$

Identity vector]

$\therefore |A - \lambda I| = 0$ is called characteristic equation.

1

The solution of characteristic eqⁿ is called eigen value.

EX-1

Find eigen value and eigen vector of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

→

Characteristic eqⁿ of matrix A is given by

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 - \lambda & 0 \\ 8 & -1 - \lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3 - \lambda & 0 \\ 8 & -1 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (3 - \lambda)(-1 - \lambda) - 0(8)$$

$$|A - \lambda I| = -3 - 3\lambda + \lambda + \lambda^2$$

$$|A - \lambda I| = \lambda^2 - 2\lambda - 3 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda^2 - 3\lambda + \lambda - 3 = 0$$

$$\lambda(\lambda - 3) + 1(\lambda - 3) = 0$$

$$\therefore (\lambda + 1)(\lambda - 3) = 0$$

$$\therefore \lambda + 1 = 0 \quad \text{or} \quad \lambda - 3 = 0$$

$$\therefore \boxed{\lambda = -1} \quad \text{or} \quad \boxed{\lambda = 3}$$

$\lambda = 3, -1$ are called eigen values.

→ (1) The eigen vector corresponding to the eigen value $\lambda = -1$ is given by

$$(A - \lambda I)X = 0$$

$$\therefore (A - (-1)I)X = 0 \quad |X = A|$$

$$\therefore (A + I)X = 0$$

$$(2) \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad |X = A|$$

[X is a column matrix

$$AX + X + X = 0 \quad |IX = A|$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$= \begin{matrix} 4x_1 \downarrow & \downarrow \text{ હોઈ} \\ 2 \times 2 & 2 \times 1 \leftarrow \text{સત્તા} \end{matrix}$$

સરખા લેવા મોડેલ

$$m \times n \quad n \times p$$

Ans: $m \times p$ સત્તા = 2×1

$$\therefore \begin{bmatrix} 4x_1 + 0x_2 \\ 8x_1 + 0x_2 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 4x_1 \\ 8x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1 = 0 \quad \text{અથ} \quad 8x_1 = 0$$

$$\boxed{x_1 = 0}$$

$$, \quad \boxed{x_1 = 0}$$

let $x_2 = t$; $t \in \mathbb{R}$

→ Eigen vector ~~is~~ $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

→ (2) The eigen vector corresponding to the eigen value $\lambda = 3$ is given by

$$(A - \lambda I)X = 0$$

$$(A - 3I)X = 0$$

$$(A - 3I)X = 0$$

$$\left(\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Ex =

$$\begin{bmatrix} 0 & 0 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So

$$\begin{bmatrix} 0x_1 + 0x_2 \\ 8x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 8x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$8x_1 - 4x_2 = 0$$

$$2x_1 - x_2 = 0$$

$$x_2 = 2x_1$$

$$\text{Let } x_1 = t, \quad x_2 = 2t; \quad t \in \mathbb{R}$$

$$\rightarrow \text{Eigen vector } x_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix}$$

$$\begin{bmatrix} t \\ 2t \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}$$

Ex-2: Find eigen value & eigen vector.

$$A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$$

Solⁿ: Characteristic eqⁿ of matrix A is given by

$$A - \lambda I = 0$$

$$A - \lambda I = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 10 - \lambda & -9 \\ 4 & -2 - \lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 10 - \lambda & -9 \\ 4 & -2 - \lambda \end{vmatrix} = 0$$

$$(10 - \lambda)(-2 - \lambda) - (-9)(4) = 0$$

$$-20 - 10\lambda + 2\lambda + \lambda^2 + 36 = 0$$

$$-20 - 8\lambda + \lambda^2 + 36 = 0$$

$$\lambda^2 - 8\lambda + 16 = 0$$

$$\lambda^2 - 4\lambda - 4\lambda + 16 = 0$$

$$\lambda(\lambda - 4) - 4(\lambda - 4) = 0$$

$$(\lambda - 4)(\lambda + 4) = 0$$

$$\therefore \boxed{\lambda = 4}, \boxed{\lambda = 4}$$

$\lambda = 4$ are called eigen values.

(1) The eigen vector corresponding to the eigen value $\lambda = 4$ is given by :

$$(A - \lambda I) X = 0$$

$$\therefore (A - (4)I) X = 0$$

$$\therefore \left(\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\therefore \left(\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 6x_1 - 9x_2 \\ 4x_1 - 6x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$6x_1 - 9x_2 = 0 \quad \text{or} \quad 4x_1 - 6x_2 = 0$$

$$2x_1 - 3x_2 = 0 \quad \text{or} \quad 2x_1 - 3x_2 = 0$$

$$2x_1 = 3x_2$$

$$x_1 = \frac{3x_2}{2}$$

$$\text{let } x_2 = t$$

$$x_1 = \frac{3t}{2}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/2 t \\ t \end{bmatrix}$$

$$= t \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

Ex-3 Find eigen value and eigen vector of $A = \begin{bmatrix} 3 & 9 \\ 2 & 0 \end{bmatrix}$

Solⁿ : Characteristic eqⁿ of matrix A is given by

$$|(A - \lambda I)| = 0$$

$$A - \lambda I = \begin{bmatrix} 3 & 9 \\ 2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & 9 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 - \lambda & 9 \\ 2 & -\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3 - \lambda & 9 \\ 2 & -\lambda \end{vmatrix}$$

$$\therefore (3 - \lambda)(-\lambda) - 18 = 0$$

$$\therefore -3\lambda + \lambda^2 - 18 = 0$$

$$\therefore \lambda^2 - 3\lambda - 18 = 0$$

$$\therefore \lambda^2 - 6\lambda + 3\lambda - 18 = 0$$

$$\therefore \lambda(\lambda - 6) + 3(\lambda - 6) = 0$$

$$\therefore (\lambda + 3)(\lambda - 6) = 0$$

$$\therefore \boxed{\lambda = -3} \text{ or } \boxed{\lambda = 6}$$

$\lambda = -3, 6$ are called eigen values.

→ (1) The eigen vector corresponding to the eigen value $\lambda = -3$ is given by

$$(A - \lambda I)X = 0$$

$$(A - (-3)I)X = 0$$

$$(A + 3I)X = 0$$

$$\left(\begin{bmatrix} 3 & 9 \\ 2 & 0 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} 3 & 9 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6x_1 + 9x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$6x_1 + 9x_2 = 0 \quad \text{or} \quad 2x_1 + 3x_2 = 0$$

$$2x_1 + 3x_2 = 0$$

$$2x_1 = -3x_2 \quad \text{or} \quad x_1 = \frac{-3}{2}x_2$$

$$x_1 = \frac{-3}{2}x_2$$

$$x_2 = t, \quad x_1 = -\frac{3t}{2}; t \in \mathbb{R}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3/2 t \\ t \end{bmatrix} = t \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$$

(2) The eigen vector corresponding to the given value $\lambda = 6$ is given by

$$(A - \lambda I) X = 0$$

$$\therefore (A - (6)I) X = 0$$

$$\therefore (A - 6I) X = 0$$

$$\therefore \left[\left(\begin{bmatrix} 3 & 9 \\ 2 & 0 \end{bmatrix} - 6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right] = 0$$

$$\therefore \left[\left(\begin{bmatrix} 3 & 9 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right] = 0$$

$$\therefore \begin{bmatrix} -3 & 9 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3x_1 + 9x_2 \\ 2x_1 - 6x_2 \end{bmatrix} = 0$$

$$-3x_1 + 9x_2 = 0$$

$$\text{or } 2x_1 - 6x_2 = 0$$

$$-x_1 + 3x_2 = 0$$

$$\text{or } x_1 - 3x_2 = 0$$

$$x_1 = 3x_2$$

$$\text{or } x_1 = 3x_2$$

$$x_2 = t$$

$$x_1 = 3t$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Q4)

Ex-(4)

Find eigen value and eigen vector of $A = \begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$

Solⁿ:-

Characteristic eqⁿ of matrix A is given by

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -5 \\ 3 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -5 \\ 3 & 2-\lambda \end{vmatrix}$$

$$\begin{aligned} \therefore (1-\lambda)(2-\lambda) - (-5)(3) &= 0 \\ 2-\lambda-2\lambda+\lambda^2+15 &= 0 \\ 2-3\lambda+\lambda^2+15 &= 0 \\ \lambda^2-3\lambda+17 &= 0 \end{aligned} \quad \text{X}$$

* $ax^2 + bx + c = 0$ જે x માં દ્વિઘાત સમીકરણ છે.

$$\Delta = b^2 - 4ac$$

$\Delta > 0 \Rightarrow$ બે ભિન્ન, વાસ્તવિક બીજ મળે.
(different & real)

$\Delta = 0 \Rightarrow$ equal & real.
બે સમાન બીજ મળે.

$\Delta < 0 \Rightarrow$ બે બીજ કૌંડર સંખ્યા મળે.
Complex numbers

Case 18 $x_1, x_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$

Case 3 $x_1, x_2 = \frac{-b \pm \sqrt{-\Delta} i}{2a}$
 discriminant
 null?

Ex 1 $x^2 - 3x + 17 = 0$

$a=1, b=-3, c=17$

$\Delta = b^2 - 4ac$

$\Delta = (-3)^2 - 4(1)(17)$

$\Delta = 9 - 68$

$\Delta = -59 < 0$

$x_1, x_2 = \frac{-b \pm \sqrt{-\Delta} i}{2a}$

$= \frac{3 \pm \sqrt{59} i}{2}$

(11-11-11)

$x_1 = \frac{3 + \sqrt{59} i}{2}$

$x_2 = \frac{3 - \sqrt{59} i}{2}$

Ex-2

$$3x^2 + 9x + 5 = 0$$

$$a = 3, \quad b = 9, \quad c = 5$$

$$\Delta = b^2 - 4ac$$

$$\Delta = (9)^2 - 4(3)(5)$$

$$\Delta = 81 - 60$$

$$\Delta = 21 > 0$$

$$x_1, x_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$x_1, x_2 = \frac{-9 \pm \sqrt{21}}{2(3)}$$

$$x_1, x_2 = \frac{-9 \pm \sqrt{21}}{6}$$

$$x_1 = \frac{-9 + \sqrt{21}}{6}, \quad x_2 = \frac{-9 - \sqrt{21}}{6}$$

Ex-4. Find the eigen values & eigen vectors of following matrix.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Solⁿ: The characteristic eqⁿ is given by

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore 4-\lambda [(1-\lambda)^2 - 0] - 0 [-2(1-\lambda) - 0] + 1 [0 - (-2)(1-\lambda)] = 0$$

$$\therefore 4-\lambda (1-\lambda)^2 + 2 - 2\lambda = 0$$

$$\therefore 4-\lambda (1-2\lambda+\lambda^2) + 2-2\lambda = 0$$

$$4 - \lambda - 8\lambda + 2\lambda^2 + 4\lambda^2 - \lambda^3 + 21\lambda - 2\lambda = 0$$

$$\therefore -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\therefore -(\lambda^3 - 6\lambda^2 + 11\lambda - 6) = 0$$

$$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$1 - 6 + 11 - 6 = 0$ બધાં સદાગુણકોને સરવાળાં 0 થાય

$\therefore$ એક અવયવ $\lambda - 1$ થશે

$\lambda - 1$ ની 0 સાથે સરખાવતાં ...

$$\lambda - 1 = 0 \quad \therefore \lambda = 1$$

	1	-6	11	-6
1	0	1	-5	6
	1	-5	6	0

$$\therefore (\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$

$$\therefore (\lambda - 1)(\lambda^2 - 3\lambda - 2\lambda + 6) = 0$$

$$\therefore \lambda = 1 \text{ ઓ } \lambda(\lambda - 3) - 2(\lambda - 3) = 0$$

$$\therefore (\lambda - 2)(\lambda - 3) = 0$$

$$\boxed{\lambda = 1}, \quad \therefore \quad \boxed{\lambda = 2}, \quad \boxed{\lambda = 3}$$

Q13

The eigen vector corresponding to the eigen value $\lambda = 1$ is given by

$$(A - \lambda I)X = 0$$

$$(A - (1)I)X = 0$$

$$(A - I)X = 0$$

$$\left(\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= 0$$

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3x_1 + x_3 \\ -2x_1 \\ -2x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_1 + x_3 = 0, \quad -2x_1 = 0, \quad -2x_1 = 0$$

$$0 = 3(0) + x_3 = 0 \quad x_1 = 0$$

$$x_3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \\ 0 \end{bmatrix} \quad \lambda \in \mathbb{R}$$

$$\lambda \in \mathbb{R}$$

(a) The eigen vector corresponding to the eigen value $\lambda = 2$ is given by

$$(A - \lambda I)X = 0$$

$$(A - 2I)X = 0$$

$$\left(\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_3 \\ -2x_1 - x_2 \\ -2x_1 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_3 = 0 ; -2x_1 - x_2 = 0 ; -2x_1 - x_3 = 0$$

$$-2x_1 = +x_3 \quad x_2 = -2x_1$$

$$x_1 \neq 0 \neq x_2 \neq x_3$$

$$x_1 = t$$

$$x_2 = -2t$$

$$x_1 \neq +2x_2$$

$$x_3 = -2t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ -2t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

$$; t \in \mathbb{R}$$

(3) The eigen vector corresponding to the eigen value $\lambda = 3$ is given by

$$(A - \lambda I)X = 0$$

$$(A - 3I)X = 0$$

$$\left(\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_3 \\ -2x_1 - 2x_2 \\ -2x_1 - 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0 ; -2x_1 - 2x_2 = 0 ; -2x_1 - 2x_3 = 0$$

$$x_1 = -x_3 ; x_1 = -x_2 ; x_1 = -x_3$$

$$\text{let } x_2 = x_3 = t$$

$$x_1 = -t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ -t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} ; t \in \mathbb{R}$$

Ex-5

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

→ The characteristic eqⁿ is given by

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$-\lambda[(2-\lambda)(3-\lambda) - 0] - 2[-(2-\lambda)]$$

$$\therefore -\lambda (6 - 2\lambda - 3\lambda + \lambda^2) + 2(2 - \lambda) = 0$$

$$\therefore -6\lambda + 2\lambda^2 + 3\lambda^2 - \lambda^3 + 4 - 2\lambda = 0$$

$$\therefore -6\lambda + 5\lambda^2 - \lambda^3 + 4 - 2\lambda = 0$$

$$\therefore -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\therefore -[\lambda^3 - 5\lambda^2 + 8\lambda - 4] = 0$$

$$\therefore \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

	1	-5	8	-4
1	0	1	-4	4
	1	-4	4	0

$$\therefore (\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$$

$$\therefore (\lambda - 1)(\lambda^2 - 2\lambda - 2\lambda + 4) = 0$$

$$\therefore (\lambda - 1)(\lambda(\lambda - 2) - 2(\lambda - 2)) = 0$$

$$\therefore (\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$$

$\therefore \lambda = 1, \lambda = 2, \lambda = 2$ are eigen values.

(1) The eigen vector corresponding to the eigen value $\lambda = 1$ is given by

$$(A - \lambda I)X = 0$$

$$(A - 1I)X = 0$$

$$(A - I)X = 0$$

$$\left(\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -x_1 - 2x_3 \\ x_1 + x_2 + x_3 \\ x_1 + 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 - 2x_3 = 0 ; x_1 + x_2 + x_3 = 0$$

$$\therefore x_1 = -2x_3 ; x_1 + 2x_3 = 0$$

$$x_1 = -2x_3$$

$$-2x_3 + x_2 + x_3 = 0$$

$$x_2 - x_3 = 0$$

$$\therefore x_2 = x_3$$

$$\text{let } x_3 = t$$

$$x_2 = t$$

$$x_1 = -2t$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix};$$

$t \in \mathbb{R}$

(2) The eigen vector corresponding to the eigen value $\lambda = 2$ is given by

$$(A - \lambda I)X = 0$$

$$(A - 2I)X = 0$$

$$\left(\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2x_1 - 2x_3 \\ x_1 + x_3 \\ x_1 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_3 = 0; \quad x_1 + x_3 = 0; \quad x_1 + x_3 = 0$$

$$x_1 = -x_3$$

~~$$\text{Let } x_2 = t_1, \quad x_1 = -t_1, \quad x_3 = t_1$$~~

~~$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t_1 \\ t_2 \\ t_1 \end{bmatrix}$$~~

~~$$= t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}; t \in \mathbb{R}$$~~

$$\text{Let } x_3 = t_1, \quad x_1 = -t_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t_1 \\ 0 \\ t_1 \end{bmatrix} = t_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$t_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

↑
જ્યાં x_1 ની મૂલ્ય ત્યારે
આવી ગણિત તરીકે ---

Ex-5

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Sol^m:-

The characteristic eqⁿ is given by

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 0 - \lambda & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8 - \lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8 - \lambda \end{vmatrix} = 0$$

$$-\lambda [-\lambda (8 - \lambda) - (-17)] - 1 (-4) = 0$$

$$-\lambda [-8\lambda + \lambda^2 + 17] + 4 = 0$$

$$8\lambda^2 - \lambda^3 - 17\lambda + 4 = 0$$

$$-\lambda^3 + 8\lambda^2 - 17\lambda + 4 = 0$$

$$-(\lambda^3 - 8\lambda^2 + 17\lambda - 4) = 0$$

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

આપણે વિચારીએ દાખલાની જેમથી

અવધાન $\lambda - 4$ થી $\lambda^3 - 8\lambda^2 + 17\lambda - 4$

$$\begin{array}{r|rrrrr} & 1 & -8 & 17 & -4 & \\ \hline \lambda - 4 & 0 & 4 & -16 & 4 & \end{array}$$

$$\begin{array}{r|rrrrr} & 1 & -4 & 1 & 0 & \\ \hline \lambda - 4 & 0 & 24 & -3 & 4 & \end{array}$$

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

$$\lambda = 4 \quad \therefore \lambda^2 - 4\lambda + 1 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, b = -4, c = 1$$

$$\lambda = \frac{+4 \pm \sqrt{16 - 4}}{2}$$

$$\lambda = 4 \pm 2\sqrt{3}$$

$$\lambda = 2 \pm \sqrt{3}$$

$$\lambda_2 = 2 + \sqrt{3}, \quad \lambda_3 = 2 - \sqrt{3}$$

(1) The eigen vectors corresponding to the eigen values $\lambda = 4$ is given by

$$(A - \lambda I)X = 0$$

$$(A - 4I)X = 0$$

$$\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 4 & -17 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4x_1 + x_2 \\ -4x_2 + x_3 \\ 4x_1 - 17x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x_1 + x_2 = 0, \quad -4x_2 + x_3 = 0$$

$$x_2 = 4x_1, \quad -4(4x_1) + x_3 = 0$$

$$-16x_1 + x_3 = 0$$

$$x_3 = 16x_1$$

$$4x_1 - 17x_2 + 4x_3 = 0$$

$$\text{Let } x_1 = t$$

$$x_2 = 4t \quad ; t \in \mathbb{R}$$

$$x_3 = 16t$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 4t \\ 16t \end{bmatrix} = t \begin{bmatrix} 1 \\ 4 \\ 16 \end{bmatrix}$$

Ex-6

$$A = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$$

Solⁿ: The characteristic eqⁿ is given by

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 5 & 4 \\ 5 & 7-\lambda & 5 \\ 4 & 5 & -2-\lambda \end{vmatrix} = 0$$

$$\therefore (-2-\lambda) [(7-\lambda)(-2-\lambda) - 25]$$

$$+ 5 [5(-2-\lambda) - 20]$$

$$+ 4 [25 - 4(7-\lambda)] = 0$$

$$(-2-\lambda) [-14 - 7\lambda + 2\lambda + \lambda^2 - 25]$$

$$+ 5 [-10 - 5\lambda - 20]$$

$$+ 4 [25 - 28 + 4\lambda] = 0$$

$$\therefore (-2-\lambda) [\lambda^2 - 5\lambda - 39]$$

$$- 5 [-5\lambda - 30] + 4 [4\lambda - 3] = 0$$

$$-2\lambda^2 + 10\lambda + 78 - \lambda^3 + 5\lambda^2 + 39\lambda + 25\lambda + 150 + 16\lambda - 12 = 0$$

$$-\lambda^3 + 3\lambda^2 + 90\lambda + 216 = 0$$

$$-(\lambda^3 - 3\lambda^2 - 90\lambda - 216) = 0$$

$$\lambda^3 - 3\lambda^2 - 90\lambda - 216 = 0$$

અહીં (-3) એ મૂલ્યો જમાવી બાજુ
0 આપે તો એક અવયવ $\lambda - (-3) = 0$

$$\lambda + 3 = 0 \Rightarrow \lambda = -3$$

	1	-3	-90	-216
-3	0	-3	+18	916
	1	-6	-72	170

$$(\lambda + 3)(\lambda^2 - 6\lambda - 72) = 0$$

$$(\lambda + 3)(\lambda - 12)(\lambda + 6) = 0$$

$$\lambda + 3 = 0 \quad \lambda(\lambda - 12) + 6(\lambda - 12) = 0$$

$$\lambda + 3 = 0 \quad \lambda - 12 = 0 \quad \lambda + 6 = 0$$

$$\lambda = -3 \quad ; \quad \lambda = -6 \quad ; \quad \lambda = 12$$

→ Eigen vector for $\lambda = 12$

$$A - \lambda I = \begin{bmatrix} 12-12 & 0 & 0 \\ 0 & 12-12 & 0 \\ 0 & 0 & 12-12 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - \lambda I)X = 0$$

$$(A - 12I)X = 0$$

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 4 & 1 & 0 & 0 \\ 5 & 7 & 5 & 0 & 1 & 0 \\ 4 & 5 & -2 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|ccc} -2 & 5 & 4 & 12 & 0 & 0 \\ 5 & 7 & 5 & 0 & 12 & 0 \\ 4 & 5 & -2 & 0 & 0 & 12 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} -14 & 5 & 4 \\ 5 & -5 & 5 \\ 4 & 5 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -14x_1 + 5x_2 + 4x_3 \\ 5x_1 - 5x_2 + 5x_3 \\ 4x_1 + 5x_2 - 14x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \text{(1)} \quad -14x_1 + 5x_2 + 4x_3 &= 0 \\ \text{(2)} \quad 5x_1 - 5x_2 + 5x_3 &= 0 \\ \text{(3)} \quad 4x_1 + 5x_2 - 14x_3 &= 0 \end{aligned}$$

By Cramer's rule;

Consider eqⁿ (1) & (2)

from (2)

$$x_1 - x_2 + x_3 = 0 \quad (4)$$

from (1) & (4)

$$-14x_1 + 5x_2 + 4x_3 = 0$$

$$x_1 - x_2 + x_3 = 0$$

$$\frac{x_1}{5+4} = \frac{-x_2}{-14-4} = \frac{x_3}{14-5}$$

આથી લીપની રીતે હો x_1 ઉપર હાથ મૂકી
 x_2 અને x_3 નિઝ્યાથક ની રીતે કરી.

નિઝ્યાથક માં $\begin{vmatrix} + & - & + \\ - & + & - \end{vmatrix}$ હોય તેથી

બીજી વૈલ્યુ = ન આવશે ;

$$\frac{x_1}{9} = \frac{-x_2}{-18} = \frac{x_3}{9}$$

$$\frac{x_1}{9} = \frac{x_2}{18} = \frac{x_3}{9}$$

(જો) ની રીતે નિઝ્યાથક

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1} = k ; k \in \mathbb{R}$$

$$x_1 = k$$

$$x_2 = 2k$$

$$(2) x_3 = k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{if } k \in \mathbb{R}$$

$\Rightarrow$ Eigen Vector for $\lambda = -3$

$$(A - \lambda I)X = 0$$

$$\left[\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= 0$$

$$\therefore \left[\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 5 & 4 \\ 5 & 10 & 5 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 5x_2 + 4x_3 \\ 5x_1 + 10x_2 + 5x_3 \\ 4x_1 + 5x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 5x_2 + 4x_3 &= 0 \quad (1) \\ 5x_1 + 10x_2 + 5x_3 &= 0 \quad (2) \\ 4x_1 + 5x_2 + x_3 &= 0 \quad (3) \end{aligned}$$

from (2)

$$x_1 + 2x_2 + x_3 = 0 \quad (4)$$

from (1) & (4)

$$\begin{aligned} x_1 + 5x_2 + 4x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0 \end{aligned}$$

$$\begin{array}{c|cc|c|cc|c} x_1 & 5 & 4 & - & 2 & 1 & 4 & - & 1 & 5 \\ x_2 & 2 & 1 & - & 1 & 1 & 1 & - & 1 & 2 \end{array}$$

$$\frac{x_1}{5-8} = \frac{-x_2}{1-4} = \frac{x_3}{-2-5}$$

$$\frac{x_1}{-3} = \frac{-x_2}{-3} = \frac{x_3}{-3}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1} = k$$

$$x_1 = -k$$

$$x_2 = k$$

$$x_3 = -k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} ; k \in \mathbb{R}$$

→ Eigen vector for $\lambda = -6$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix} - (-6) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 5 & 4 \\ 5 & 13 & 5 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4x_1 + 5x_2 + 4x_3 \\ 5x_1 + 13x_2 + 5x_3 \\ 4x_1 + 5x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 4x_1 + 5x_2 + 4x_3 &= 0 \quad (1) \\ 5x_1 + 13x_2 + 5x_3 &= 0 \quad (2) \\ 4x_1 + 5x_2 + 4x_3 &= 0 \quad (3) \end{aligned}$$

from (1) & (2)

$$\begin{aligned} x_1 &= -x_2 + x_3 \\ \begin{vmatrix} 5 & 4 \\ 5 & 4 \end{vmatrix} &= \begin{vmatrix} 4 & 4 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 4 & 5 \end{vmatrix} \\ 20 - 20 &= 16 - 16 \end{aligned}$$

$$\begin{aligned} x_1 &= -x_2 + x_3 \\ \begin{vmatrix} 5 & 4 \\ 13 & 5 \end{vmatrix} &= \begin{vmatrix} 4 & 4 \\ 5 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 5 & 13 \end{vmatrix} \\ 25 - 52 &= 20 - 20 = 52 - 25 \end{aligned}$$

$$\begin{aligned} x_1 &= -x_2 + x_3 \\ 25 - 52 &= 20 - 20 = 52 - 25 \end{aligned}$$

$$\begin{aligned} x_1 &= -x_2 + x_3 \\ -27 &= 0 = 27 \end{aligned}$$

$$\begin{aligned} x_1 &= -x_2 + x_3 = k \\ -1 &= 0 = 1 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad ; k \in \mathbb{R}$$

$$C = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + -1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C + K(8) = (-1, 8) + (-1, 8) = (-2, 16)$$

$$D = (1, 1)$$

Ex-7 Find eigen values & Find eigen vectors

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Solⁿ: The characteristic eqⁿ is given by $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(3-\lambda)(3-\lambda) - (1-\lambda)(3-\lambda) - (1-\lambda)(3-\lambda) = 0$$

$$0 = (3-\lambda)(3-\lambda)(3-\lambda) - (1-\lambda)(3-\lambda) - (1-\lambda)(3-\lambda)$$

$$3 - \lambda (3 - \lambda^2 - 1) = 1 (3 - \lambda - (-1))$$

$$+ 1 (-1 - (3 - \lambda)) = 0$$

$$3 - \lambda (4 - 6\lambda + \lambda^2 - 1) - (3 - \lambda + 1)$$

$$+ (-1 - 3 + \lambda) = 0$$

$$3 - \lambda (\lambda^2 - 6\lambda + 8) - (-\lambda + 4)$$

$$+ (\lambda - 4) = 0$$

$$3\lambda^2 - 18\lambda + 24 - \lambda^3 + 6\lambda^2 - 8\lambda + \lambda - 4$$

$$+ \lambda - 4 = 0$$

$$-\lambda^3 + 9\lambda^2 - 24\lambda + 16 = 0$$

$$-(\lambda^3 - 9\lambda^2 + 24\lambda - 16) = 0$$

$$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$$

1	-9	24	-16
0	1	-8	16
1	-8	16	0

$$(\lambda - 1)(\lambda^2 - 8\lambda + 16) = 0$$

$$(\lambda - 1)(\lambda^2 - 4\lambda - 4\lambda + 16) = 0$$

$$(\lambda - 1)(\lambda(\lambda - 4) - 4(\lambda - 4)) = 0$$

$$(\lambda - 1)(\lambda - 4)(\lambda - 4) = 0$$

$\lambda = 1$; $\lambda = 4$; $\lambda = 4$ are eigen values.

(1A) Eigen value of $\lambda = 1$

$$(A - \lambda I)X = 0 \quad \therefore (A - (1)I)X = 0$$

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & -1 & 0 & 1 & 0 \\ 1 & -1 & 3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{-(1)} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & -1 & 0 & 1 & 0 \\ 1 & -1 & 3 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{l} 2x_1 + x_2 + x_3 \\ x_1 + 2x_2 - x_3 \\ x_1 - x_2 + 2x_3 \end{array} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_2 + x_3 = 0 \quad \text{--- (1)}$$

$$x_1 + 2x_2 - x_3 = 0 \quad \text{--- (2)}$$

$$x_1 - x_2 + 2x_3 = 0 \quad \text{--- (3)}$$

By Cramer's rule

from (1) & (2)

$$x_1 = -x_2 = x_3$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 1 \end{array} \right] \quad \left[\begin{array}{cc|cc} 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right]$$

$$\begin{aligned} x_1 &= -x_2 - x_3 \\ -1-2 & \quad -3x_2 \quad -4x_3 \end{aligned}$$

$$\therefore x_1 = -3x_2 - 4x_3 \quad x_2 = k, x_3 = k$$

$$x_1 = -3k, x_2 = +3k, x_3 = 3k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3k \\ 3k \\ 3k \end{bmatrix} = 3 \begin{bmatrix} -k \\ k \\ k \end{bmatrix}$$

$$= k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, k \in \mathbb{R}$$

(2) Eigen value: $\lambda = 4$

$$(A - \lambda I)X = 0$$

$$(A - 4I)X = 0$$

$$\left[\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -x_1 + x_2 + x_3 \\ x_1 - x_2 - x_3 \\ x_1 - x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 + x_3 = 0 \quad \text{--- (1)}$$

$$x_1 - x_2 - x_3 = 0 \quad \text{--- (2)}$$

$$x_1 - x_2 - x_3 = 0 \quad \text{--- (3)}$$

આથી ત્રણેય eqⁿ સમાન છે તો ગમે
તે કયો કયો ૦ ધારણું.

$$\text{Suppose } x_2 = 0$$

from (2)

$$x_1 - x_3 = 0$$

$$\therefore x_1 = x_3 = \lambda$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda \in \mathbb{R}$$

Ex-8

$$A = \begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$$

Solⁿ:

The characteristic eqⁿ is given by

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 7-\lambda & -2 & 1 \\ -2 & 10-\lambda & -2 \\ 1 & -2 & 7-\lambda \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 7-\lambda & -2 & 1 \\ -2 & 10-\lambda & -2 \\ 1 & -2 & 7-\lambda \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 7-\lambda & -2 & 1 \\ -2 & 10-\lambda & -2 \\ 1 & -2 & 7-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & -2 & 1 \\ -2 & 10-\lambda & -2 \\ 1 & -2 & 7-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} & (7-\lambda) [(10-\lambda)(7-\lambda) - (4)] \\ & + 2 [-2(7-\lambda) - (-20)] \\ & - 1 [-4 - (10-\lambda)2] \end{aligned}$$

$$\begin{aligned} & (7-\lambda) [70 - 10\lambda - 7\lambda + \lambda^2 - 4] \\ & + 2 [-14 + 2\lambda + 20] \\ & + 1 [-4 - 10 + \lambda] \end{aligned}$$

$$7\lambda^2 - 119\lambda + 462 - \lambda^3 + 17\lambda^2 - 66\lambda + 4\lambda - 24 + \lambda = 6 = 0$$

$$- \lambda^3 + 24\lambda^2 - 180\lambda + 432 = 0$$

$$- (\lambda^3 - 24\lambda^2 + 180\lambda - 432) = 0$$

$$\lambda^3 - 24\lambda^2 + 180\lambda - 432 = 0$$

$$\lambda^3 - 24\lambda^2 + 180\lambda - 432 = 0$$

$$\lambda^3 - 24\lambda^2 + 180\lambda - 432 = 0$$

$$\lambda^3 - 24\lambda^2 + 180\lambda - 432 = 0$$

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$$\lambda^3 - 24\lambda^2 + 180\lambda - 432 = 0$$

$$\lambda^3 - 24\lambda^2 + 180\lambda - 432 = 0$$

(1) eigen value for $\lambda = 12$

$$(A - \lambda I) X = 0$$

$$(A - 12I) X = 0$$

$$\begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix} \sim \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix} \sim \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -2 & 1 \\ -2 & -2 & -2 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5x_1 - 2x_2 + x_3 \\ -2x_1 - 2x_2 - 2x_3 \\ x_1 - 2x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -5x_1 - 2x_2 + x_3 &= 0 \quad \text{--- (1)} \\ -2x_1 - 2x_2 - 2x_3 &= 0 \quad \text{--- (2)} \\ x_1 - 2x_2 + 5x_3 &= 0 \quad \text{--- (3)} \end{aligned}$$

from (1) & (2)

By Cramer's rule

$$\frac{x_1}{4+2} = \frac{x_2}{10+2} = \frac{x_3}{10-2}$$

$$\frac{x_1}{6} = \frac{-x_2}{6} = \frac{x_3}{8}$$

$$x_1 = -x_2 + x_3 = k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$k \in \mathbb{R}$$

(2)

Eigen value for $\lambda = 6$

$$(A - \lambda I)X = 0$$

$$(A - 6I)X = 0$$

$$\left[\begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix} - 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 2x_2 + x_3 = 0 \quad - (1)$$

$$-2x_1 + 4x_2 - 2x_3 = 0 \quad - (2)$$

$$x_1 - 2x_2 + x_3 = 0 \quad - (3)$$

By cramer's rule

from (1) & (2)

$$\begin{array}{ccc} x_1 & - & x_2 & & x_3 \\ 4-4 & & 2-2 & & 4-4 \end{array}$$

let us suppose $x_2 = 0$

put the value of x_2 in (1)

$$x_1 - 2x_2 + x_3 = 0$$

$$\therefore x_1 - 0 + x_3 = 0 \Rightarrow x_1 + x_3 = 0$$

$$x_1 + x_3 = 0$$

$$x_1 = -x_3$$

let $x_3 = t$ $\therefore x_1 = -t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$

$$(1) \Rightarrow x_1 - 2x_2 + x_3 = 0$$

$$(2) \Rightarrow x_1 + x_2 - x_3 = 0$$

$$(3) \Rightarrow x_1 + x_2 + x_3 = 0$$

Properties of eigen values:

✓ (1)

If A is real matrix its eigen values are real numbers or complex conjugate pair $z = a+bi$, $\bar{z} = a-bi$

(2)

The product of eigen value of matrix A is equal to determinant of matrix A . [ଅନୁଗୁଣ ନିମ୍ନାଂଶ]

Suppose $\lambda_1, \lambda_2, \lambda_3$ are eigen values.

$$\therefore \lambda_1 \lambda_2 \lambda_3 = |A|$$

Suppose λ_1, λ_2 are eigen values

$$\therefore \lambda_1 \lambda_2 = |A|$$

(3)

The square matrix A & A^T have same eigen value.

ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

$\therefore A$ and A^T have same eigen value.

4)

If all the eigen values are non zero for any square matrix A

singular

if $\lambda_1, \lambda_2, \dots, \lambda_n = 0$ then A is also singular

$[A]_{n \times n}$, $\lambda_1, \lambda_2, \dots, \lambda_n \neq 0$ non-singular matrix

any

$$\therefore \det(A) = 0$$

If then A is non singular matrix ($|A| \neq 0$)

Otherwise it is singular matrix

(5) If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values of matrix A then $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots + \frac{1}{\lambda_n}$ are the eigen values for A^{-1} .

(6) If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values of matrix A then $k\lambda_1, k\lambda_2, \dots, k\lambda_n$ are eigen values.

(7) If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values of matrix A then $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$ are eigen values of matrix A^m .
 $A \rightarrow 1, 2 \quad A^7 \rightarrow 1^7, 2^7$

(8) The eigen values of symmetric matrix ($A^T = A$) are real and equal.

Ex: For example find the eigen values of

$$A = A^T$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

→

find eigen value of A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 4 & 8 \end{bmatrix}$$

* Diagonalization is

A square matrix A is called diagonalizable if there is invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

The matrix P is said to be diagonalize A.

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 4 & 8 \end{bmatrix}$$

* Theorem:

If A is an $n \times n$ matrix then the following are equivalent

(a) A is diagonalizable

(b) A has linearly independent eigen vectors.

* Procedure for diagonalizing

a matrix:

Step-1: Find n linearly independent eigen vectors of A say $X_1, X_2, \dots, X_n$

Step-2: Form the matrix P having $X_1, X_2, \dots, X_n$ as its column vectors

For ex:

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Eigen values}$$

Step-3 : The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2, \dots, \lambda_n$ as its successive diagonal entries

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(1) Find the matrix P that diagonalizes A and determine $P^{-1}AP$

$$A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

Solⁿ: The characteristic eqⁿ is given by

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix} = 0$$

~~$$(1-\lambda)^2 - 0 = 0$$~~

~~$$1 - 2\lambda + \lambda^2 = 0$$~~

~~$$\lambda^2 - 2\lambda + 1 = 0$$~~

~~$$\lambda^2 - \lambda - \lambda + 1 = 0$$~~

~~$$\therefore \lambda(\lambda-1)$$~~

$$(1-\lambda)(1-\lambda) = 0$$

$$1 - \lambda + \lambda + \lambda^2 = 0$$

$$\lambda^2 - 1 = 0$$

$$\therefore \lambda^2 = 1$$

$$\therefore \boxed{\lambda = \pm 1}$$

(1) For eigen value $\lambda = 1$

$$(A - \lambda I)X = 0$$

$$(A - 1I)X = 0$$

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] - \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$6x_1 - 2x_2 = 0$$

$$3x_1 - x_2 = 0$$

$$x_2 = 3x_1$$

$$x_1 = k_1$$

$$x_2 = 3k_1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$k_1 \in \mathbb{R}$

(2) For eigen values $\lambda = -1$

$$(A - \lambda I)x = 0$$

$$(A + I)x = 0$$

$$\left[\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ 6x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \text{adj } P$$

Let $x_1 = 0$
 $x_2 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} ; 1 \in \mathbb{R}$$

$$P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$|P| = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}$$

$$|P| = 1 - 0$$

$$|P| = 1 \neq 0$$

$\therefore P^{-1}$ exist

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{matrix} 4-2 \\ 3-1 \end{matrix}$$

(2×2 માટે P^{-1} શોધવાની જ રીત છે $\rightarrow 1$)
 $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ (1 & 0 નું સ્થાન બદલો)
 (0 અને 3 સ્થાન જ રહે
 માત્ર નિશાની બદલો)

$$P^{-1} A P = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$P^{-1}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

$$P^{-1}A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(2) $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

Solⁿ:- The characteristic eqⁿ is given that

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix} = A - \lambda I$$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$-1-\lambda [(4-\lambda)(3-\lambda) - 0] - 4 [(-3)(3-\lambda) - 0] - 2 [-3 - (-3)(4-\lambda)] = 0$$

$$-1-\lambda [12 - 4\lambda - 3\lambda + \lambda^2] - 4 [-9 + 3\lambda] - 2 [-3 - (-12 + 3\lambda)] = 0$$

$$-1-\lambda [\lambda^2 - 7\lambda + 12] - 4 [3\lambda - 9] - 2 [-3 + 12 - 3\lambda] = 0$$

$$-\lambda^2 + 7\lambda - 12 - \lambda^3 + 7\lambda^2 - 12\lambda - 12\lambda + 36 - 18 + 6\lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\begin{array}{c|cccc} & 1 & -6 & 11 & -6 \\ 1 & 0 & 1 & -5 & 6 \\ & 1 & -5 & 6 & 0 \end{array}$$

$$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda - 1)(\lambda^2 - 3\lambda - 2\lambda + 6) = 0$$

$$(\lambda - 1) [\lambda(\lambda - 3) - 2(\lambda - 3)] = 0$$

$$\lambda = 1, \lambda = 3, \lambda = 2$$

(1) The eigen value for $\lambda = 1$

$$(A - \lambda I)X = 0$$

$$(A - 1(I))X = 0$$

$$(A - I)X = 0$$

$$\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-2x_1 + 4x_2 - 2x_3 = 0 \quad \text{--- (1)}$$

$$-3x_1 + 3x_2 = 0 \quad \text{--- (2)}$$

$$-3x_1 + x_2 + 2x_3 = 0 \quad \text{--- (3)}$$

from (1) & (3)

$$\frac{x_1}{8+2} = \frac{-x_2}{-4-6} = \frac{x_3}{-2+12}$$

$$\frac{x_1}{10} = \frac{-x_2}{-10} = \frac{x_3}{10}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = k$$

$$x_1 = x_2 = x_3 = k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};$$

$$k \in \mathbb{R}$$

(2) The eigen value for $\lambda = 2$

$$(A - \lambda I)X = 0$$

$$(A - 2I)X = 0$$

$$\left[\begin{bmatrix} -1 & 4 & -2 \\ -3 & -4 & 0 \\ -3 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 & 4 & -2 \\ -3 & 2 & 0 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-3x_1 + 4x_2 - 2x_3 = 0 \quad \text{--- (4)}$$

$$-3x_1 + 2x_2 = 0 \quad \text{--- (5)}$$

$$-3x_1 + x_2 + x_3 = 0 \quad \text{--- (6)}$$

from (4) & (6)

$$\frac{x_1}{4+2} = \frac{-x_2}{-3+6} = \frac{x_3}{-3+12}$$

$$\frac{x_1}{6} = \frac{-x_2}{-3+9} = \frac{x_3}{9}$$

$$\frac{x_1}{2} = \frac{+x_2}{3} = \frac{x_3}{3} = k$$

$$x_1 = 2k, \quad x_2 = 3k, \quad x_3 = 3k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ 3k \\ 3k \end{bmatrix} = k \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix};$$

$$k \in \mathbb{R}$$

The eigen values for $\lambda = 3$

$$(A - \lambda I) X = 0$$

$$(A - 3I) X = 0$$

$$\left[\begin{bmatrix} -1 & 4 & -2 \\ -3 & -4 & 0 \\ -3 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -4 & 4 & -2 \\ -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-4x_1 + 4x_2 - 2x_3 = 0 \quad \text{--- (7)}$$

$$-3x_1 + x_2 + 0x_3 = 0 \quad \text{--- (8)}$$

$$-3x_1 + x_2 + 0x_3 = 0 \quad \text{--- (9)}$$

(6-7) (8-9) (7) & (8)

$$\frac{x_1}{0+6} = \frac{-x_2}{0-6} = \frac{x_3}{-4+12}$$

$$\frac{x_1}{2} = \frac{-x_2}{-6} = \frac{x_3}{8}$$

$$\frac{x_1}{2} = \frac{x_2}{6} = \frac{x_3}{8}$$

$$\frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{4} = k$$

$$x_1 = k, \quad x_2 = 3k, \quad x_3 = 4k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 3k \\ 4k \end{bmatrix} = k \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$k \in \mathbb{R}$$

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|P| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 1(12-9) - 2(4-3) + 1(3-3)$$

$$= 3 - 2$$

$$|P| = 1 \neq 0$$

P is invertible

$$P^{-1} = \frac{1}{|P|} \text{adj } P$$

Cofactors of P = $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$
(P ની અવલંબનાં ચિહ્નો)

$$P_c = \begin{bmatrix} 12-9 & -(4-3) & 3-3 \\ -(8-3) & 4-1 & -(3-2) \\ 6-3 & +(3-1) & 3-2 \end{bmatrix}$$

$$P_c = \begin{bmatrix} 3 & -1 & 0 \\ -5 & 3 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\text{adj } P = (P_c)^T = \begin{bmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

(પ્રત્યેક હાર પ્રત્યેક સ્તંભમાં ફરી)

$$P^{-1} = \begin{bmatrix} 1 & 3 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$P^{-1}AP$$

$$P^{-1}A = \begin{bmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3+15-9 & 12-20+3 & -6+0+9 \\ -1-9+6 & -4+12-2 & 2+0-6 \\ 0+3-3 & 0+4+1 & 0+0+3 \end{bmatrix}$$

$$P^{-1}A = \begin{bmatrix} 3 & -5 & 3 \\ -2 & 4 & -4 \\ 0 & -3 & 3 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 3 & -5 & 3 \\ -2 & 4 & -4 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3-5+3 & 6-15+9 & 3-15+12 \\ -2+4-4 & -4+18-12 & -2+14-16 \\ 0+3+3 & 0+9+9 & 0+9+12 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(8)

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Soln:-

The characteristic eqⁿ is given that $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= [2-\lambda] [(3-\lambda)^2 - 0] - 2[0]$$

$$(2-\lambda)(9 - 6\lambda + \lambda^2) = 0$$

$$(2-\lambda)(\lambda^2 - 6\lambda + 9) = 0$$

$$2\lambda - 12\lambda + 18 - \lambda^3 + 6\lambda^2 - 9\lambda = 0$$

$$= 0$$

$$= \lambda^3 + 8\lambda^2 - 21\lambda + 14 = 0$$

$$\lambda^3 + 8\lambda^2 - 21\lambda + 14 = 0$$

$$\begin{array}{c|ccc} & 1 & -8 & 21 & -14 \\ \lambda & 0 & 2 & -14 & 14 \\ & 1 & -6 & 7 & 0 \end{array}$$

$$(\lambda - 2)(\lambda^2 - 6\lambda + 7) = 0$$

$$(\lambda - 2)(\lambda^2 - 3\lambda - 3\lambda + 7) = 0$$

$$(\lambda - 2)[\lambda(\lambda - 3) - 3(\lambda - 3)] = 0$$

$$[\lambda - 2][\lambda - 3][\lambda - 3] = 0$$

$$\lambda = 2, \lambda = 3, \lambda = 3$$

(1) Eigen value for $\lambda = 2$

$$(A - \lambda I)X = 0$$

$$(A - 2I)X = 0$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 7 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -2 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 2 \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0$$

$$-2x_3 = 0$$

$$x_3 = 0$$

$\therefore$ let $x \in \mathbb{R}, x_1 = xk$

$$\text{let } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

$$k \in \mathbb{R}$$

(2) The eigen values for $\lambda = 3$

$$(A - \lambda I)X = 0$$

$$(A - 3I)X = 0$$

$$\left[\begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 - 2x_3 = 0$$

$$-x_1 = 2x_3$$

$$x_1 = -2x_3$$

$$\text{let } x_3 = k_1$$

$$x_1 = -2k_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k_1 \\ 0 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$k_1 \in \mathbb{R}$$

$$\rightarrow P = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$|P| = \begin{vmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1(0-0) + 2(0-0) - 2(0-0)$$

$$|P| = 0$$

P is not invertible

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2R_1 \\ 0 \\ R_2 \end{bmatrix}$$

$$= K \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = K \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$|P| = 1(0 \cdot 1) + 2(0 \cdot 0) - 1 \neq 0$$

P is invertible

$$P^{-1} = \frac{1}{|P|} \text{adj } P$$

$$\text{adj } P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & -1 & 0 \end{bmatrix}$$

$$P^T = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1}A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(4) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonalizable.

Here A is lower triangular matrix.

Solⁿ: The characteristic eqⁿ is given that $|A - \lambda I| = 0$.

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -3 & 5 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -3 & 5 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \cdot [(2-\lambda)^2 - 10] = 0$$

$$(1-\lambda)(2-\lambda)(2-\lambda) = 0$$

$$\therefore \lambda = 1, \lambda = 2, \lambda = 2$$

(1) Eigen vectors for $\lambda = 1$

$$(A - \lambda I)X = 0$$

$$(A - (1)I)X = 0$$

$$\left[\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ x_1 + x_2 \\ -3x_1 + 5x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 &= 0 & \text{--- (1)} \\ -3x_1 + 5x_2 + x_3 &= 0 & \text{--- (2)} \end{aligned}$$

from (1) $x_1 = -x_2$ --- (3)

from (2) & (3)

$$\begin{aligned} -3(-x_2) + 5x_2 + x_3 &= 0 \\ +8x_2 + x_3 &= 0 \end{aligned}$$

$$\therefore x_3 = -8x_2$$

$$x_2 = k_1$$

$$x_1 = -k_1$$

$$x_3 = -8k_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \\ -8k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ -8 \end{bmatrix} \quad \forall k_1 \in \mathbb{R}$$

(2) Eigen vector for $\lambda = 2$:

$$(A - \lambda I)X = 0$$

$$(A - 2I)X = 0$$

$$\left[\begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ -3 & 5 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} -x_1 \\ x_1 \\ -3x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0, \quad -3x_1 + 5x_2 = 0$$

$$\therefore x_2 = 0$$

Let $x_3 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad 1 \in \mathbb{R}$$

Here we have only two LI
Eigen vectors.
So, A is not diagonalizable.

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

(5)

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Matrix A is upper triangular matrix.

Example of diagonalization is given
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$\therefore$ Hence eigen value for
 $\lambda = 2, 2, 3, 3$

(1)

For eigen vector $\lambda = 2$

$$(A - \lambda I)X = 0$$

$$(A - 2I)X = 0$$

$$\left[\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 + x_4 = 0 \quad \text{--- (1)}$$

$$x_3 - x_4 = 0 \quad \text{--- (2)}$$

$$x_3 + 3x_4 = 0 \quad \text{--- (3)}$$

$$x_4 = 0 \quad \text{--- (4)}$$

$$x_4 = 0$$

from (1), (2), (3) and (4)

$$x_2 = x_3 = x_4 = 0$$

$$\text{let } x_1 = k_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad ; k_1 \in \mathbb{R}$$

(2) $\lambda = 3$ for eigen vector

$$(A - \lambda I)X = 0$$

$$(A - 3I)X = 0$$

$$\left[\begin{array}{cccc|cccc} 2 & 1 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 2 & 1 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \end{array} \right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_4 + x_2 + x_4 = 0 \quad \text{--- (1)}$$

$$-x_2 + x_3 - x_4 = 0 \quad \text{--- (2)}$$

$$2x_4 = 0 \quad \text{--- (3)}$$

from (3)

$$x_4 = 0$$

from (1)

$$x_3 = x_2, \quad x_4 = x_2$$

$$x_1 = x_2 = x_3 = x_4 = k_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_2 \\ k_2 \\ k_2 \\ 0 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad k_2 \in \mathbb{R}$$

Here we have only two LI eigen vectors so A is not diagonalizable.

Q1)

Determine whether the eigen
given matrix are diagonalizable.

C1)

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$$

Solⁿ

The characteristic eqⁿ is given by
 $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -3 \\ 1 & -1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -3 \\ 1 & -1-\lambda \end{vmatrix}$$

$$(2-\lambda)(-1-\lambda) + 3 = 0$$

$$-2 - 2\lambda + \lambda + \lambda^2 + 3 = 0$$

$$\lambda^2 - \lambda + 1 = 0$$

which is quadratic eqⁿ in λ

$$a = 1, \quad b = -1, \quad c = 1$$

$$\Delta = b^2 - 4ac$$

$$= 1 - 4$$

$$\Delta = -3 < 0$$

Therefore we have two complex conjugate roots

$$\lambda_1, \lambda_2 = \frac{-b \pm \sqrt{-\Delta}}{2a}$$

$$\lambda_1, \lambda_2 = \frac{-(-1) \pm \sqrt{-(-3)}}{2(1)}$$

$$\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{3}i}{2}$$

$$\lambda_1 = \frac{1 + \sqrt{3}i}{2}, \quad \lambda_2 = \frac{1 - \sqrt{3}i}{2}$$

Here we have two different eigen values. So for each eigen value we have eigen vector. matrix A is diagonalizable

$$(2) \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

Solⁿ

$$\lambda = 2, 2$$

A is

$$(A - \lambda I)X = 0$$

$$(A - 2I)X = 0$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 = 0, x_2 = k$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k \begin{bmatrix} 0 \\ 1 \end{bmatrix}; k \in \mathbb{R}$$

Here eigen values are two but we found only one eigen vector so A is not diagonalizable

Theorem

Statement: Cayley-Hamilton thm

Every square matrix A satisfies its own characteristic eqⁿ i.e.

$$\text{if } \lambda^n + (c_{n-1})\lambda^{n-1} + (c_{n-2})\lambda^{n-2} + \dots + c_1\lambda + c_0 = 0$$

is the characteristic eqⁿ of A then,

$$A^n + c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + \dots + A c_1 + c_0 I = 0$$

where I is the unit matrix of order n & $c_{n-1}, c_{n-2}, c_{n-3}, \dots, c_1, c_0$ are constants.

Ex-1

Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ hence find A^2, A^3, A^4, A^5

Solⁿ: The characteristic eqⁿ is given by $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 6 \\ 1 & 2-\lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 6 \\ 1 & 2-\lambda \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 6 \\ 1 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 6 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda) - 6 = 0$$

$$6 - 3\lambda - 2\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 5\lambda = 0$$

By Cayley-Hamilton theorem,

$$A^2 - 5A = 0$$

$$A^2 = 5A$$

$$(i) \quad A^2 = 5A = 5 \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix}$$

$$(ii) \quad A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 45 + 30 & 90 + 60 \\ 15 + 10 & 30 + 20 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 75 & 150 \\ 25 & 50 \end{bmatrix}$$

$$(iii) \quad A^4 = A^2 \cdot A^2$$
$$= \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix}$$

$$(iv) A^5 = A^4 \cdot A$$

$$= \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix}$$

$$(iii) A^4 = A^2 \cdot A^2$$

$$\begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 15 & 30 \\ 5 & 10 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 375 & 750 \\ 125 & 250 \end{bmatrix}$$

$$(iv) A^5 = A^4 \cdot A$$

$$= \begin{bmatrix} 375 & 750 \\ 125 & 250 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 1875 & 3750 \\ 625 & 1250 \end{bmatrix}$$

$$O = I - A^5 = A$$

Ex-2

Using Cayley Hamilton theorem find A^{-1} , A^{-2} , A^{-3} for

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Solⁿ:-

The characteristic eqⁿ is given by $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda)^2 - 2 = 0$$

$$1 - 2\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - 2\lambda - 1 = 0$$

By Cayley Hamilton theorem

$$A^2 - 2A - I = 0$$

(Cayley Hamilton theorem) I ने
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multiplying by A^{-1} on both the side,

$$(A^2)A^{-1} - 2(A)A^{-1} + I(A^{-1}) = 0$$

$$A - 2I = A^{-1} \quad (\because x x^{-1} = I)$$

$$A - 2I - A^{-1} = 0 \quad \text{--- (1)}$$

$$A^{-1} = A - 2I$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

multiplying (i) $\times A^{-1}$ both side

$$(A)A^{-1} - 2I(A^{-1}) - (A^{-1})A^{-1} = 0$$

$$I - 2A^{-1} - A^{-2} = 0 \quad \text{--- (ii)}$$

$$A^{-2} = I - 2A^{-1}$$

$$A^{-2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

multiplying (ii) with AA^{-1}

$$I(A^{-1}) = 2(A^{-1})(A^{-1}) - (A^{-2})A^{-1} = 0$$

$$A^{-1} - 2A^{-2} - A^{-3} = 0$$

$$A^{-3} = A^{-1} - 2A^{-2}$$

$$A^{-3} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} - 2 \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 6 & -8 \\ -4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 6 & -8 \\ -4 & 6 \end{bmatrix}$$

$$A^{-3} = \begin{bmatrix} -7 & 10 \\ 5 & -7 \end{bmatrix}$$

Ex-3

Verify Cayley-Hamilton theorem

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

Solⁿ - The characteristic eqⁿ is given by

$$|A - \lambda I| = 0$$

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

$$= -\lambda [-\lambda(3-\lambda) - (-3)] + 1 [0-1] = 0$$

$$= -\lambda [-3\lambda + \lambda^2 + 3] + 1 = 0$$

$$3\lambda^2 - \lambda^3 - 3\lambda + 1 = 0$$

$$-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

By Cayley-Hamilton theorem

$$A^3 - 3A^2 + 3A - I = 0 \quad (1)$$

$$\rightarrow A^2 = A \cdot A$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -3 & 3 \\ 3 & -8 & 6 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & -3 & 3 \\ 3 & -8 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -8 & 6 \\ 6 & -15 & 10 \end{bmatrix}$$

Put the all value in (1)

$$LHS = A^3 - 3A^2 + 3A - I$$

$$= \begin{bmatrix} 1 & -3 & 3 \\ 3 & -8 & 6 \\ 6 & -15 & 10 \end{bmatrix} - 3 \begin{bmatrix} 0 & 0 & 1 \\ 1 & -3 & 3 \\ 3 & -8 & 6 \end{bmatrix}$$

$$+ 3 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 3 \\ 3 & -8 & 6 \\ 6 & -15 & 10 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 3 \\ 3 & -9 & 9 \\ 9 & -24 & 18 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & -9 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \text{RHS}$$

Hence, Cayley hamilton thm verified

The Quadratic forms are defined as a homogeneous polynomial of second degree in any number of variable.

Ex. 12 $ax^2 + 2hxy + by^2$

↑

homogeneous

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અથવા	ઉ

$$(a) \quad ax^2 + by^2 + cz^2 + 2hxy + 2yz + 2zx$$

$$(3) \quad ax^2 + by^2 + cz^2 + d\omega^2 + 2hxyz + 2gxy + 2fzx + 2dxc + 2myw + 2mzw$$

In general a quadratic form in ~~an~~ n independent variables $x_1, x_2, \dots, x_n$ is a homogeneous 2nd degree polynomial of the form $Q(x) = Q(x_1, x_2, \dots, x_n)$

$$= a_{11}x_1^2 + a_{12}x_1x_2 + \dots + a_{1n}x_1x_n + a_{21}x_2x_1 + a_{22}x_2^2 + \dots + a_{2n}x_2x_n$$

$$Q(x) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + \dots + a_{nn}x_n^2$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$$

defining the vector $x = [x_1, x_2, \dots, x_n]^T$

$x = [x_1, x_2, \dots, x_n]^T$ & the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

The quadratic form $Q(x)$ may be written in the form

$$Q(x) = x^T A x$$

Here the matrix A is referred as the coefficient matrix or matrix of the quadratic form, and determinant of A is called discriminant.

The discriminant of a quadratic form

So, generalizing (1)

$$(2) \quad \dots + (a_{1n} + a_{n1}) x_1 x_n + \dots$$

$$a_{22} x_2^2 + (a_{23} + a_{32}) x_2 x_3 + \dots + (a_{2n} + a_{n2}) x_2 x_n$$

... all similar

$$a_{nn} x_n^2$$

$\therefore$ Every linear common
order 2 polynomial

In general we say that

a_{ij} & a_{ji} are both
coefficient of $x_i x_j$

$$\therefore (a_{ie} + a_{ei}) x_i x_e$$

$\uparrow \quad \uparrow \quad \uparrow$
 $x_i x_j \quad x_j x_i \quad x_i x_j$

$\therefore$ If $i \neq j$ as shown
into (2), the coefficient of
the term $x_i x_j$ is $(a_{ij} + a_{ji})$

By defining new coefficients
 c_{ij} & c_{ji} for $x_i x_j$ and $x_j x_i$

$$c_{ij} + c_{ji}$$

respectively

Such that $C_{ij} = C_{ji} = \frac{1}{2} [a_{ij} + a_{ji}]$

& $C_{ij} + C_{ji} = a_{ij} + a_{ji}$

Thus the quadratic form can be represented as

$$Q = \sum_{i=1}^n \sum_{j=1}^n C_{ij} x_i x_j$$

Where C is $C = \frac{1}{2} (A + A')$

is a real symmetric matrix

Ex:- $X'AX = [x_1, x_2] \begin{bmatrix} 2 & 4 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

1×2 2×2 2×1

$$[x_1, x_2] \begin{bmatrix} 2 & 4 \\ 8 & 1 \end{bmatrix} = 2x_1$$

$$= [2x_1 + 8x_2 \quad 4x_1 + x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2x_1^2 + 8x_1x_2 + 4x_1x_2 + x_2^2$$

$$= 2x_1^2 + 12x_1x_2 + x_2^2$$

Here $4+8=12 \Rightarrow 6+6$

Q. 10.10

$$[1 \ 2 \ 3 \ 4 \ 5 \ 6] = [i \ j]$$

12 ના અડધા
અથવા 6 ના 2

for the corresponding symmetric matrix

$$C_{ij} = \frac{1}{2} (C_{ji} + C_{ij})$$

Thus,

$$\frac{1}{2} (4+8) = 6$$

6

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$C_{11} = 2, \quad C_{12} = C_{21} = 6, \quad C_{22} = 1$$

$$C = \begin{bmatrix} 2 & 6 \\ 6 & 1 \end{bmatrix}$$

Now we get the same result as

$$X'CX = [x_1, x_2] \begin{bmatrix} 2 & 6 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$[x_1, x_2] \begin{bmatrix} 2 & 6 \\ 6 & 1 \end{bmatrix} = [2x_1 + 6x_2, 6x_1 + x_2]$$

$$[2x_1 + 6x_2, 6x_1 + x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$2x_1^2 + 6x_1x_2 + 6x_1x_2 + x_2^2$$

$$2x_1^2 + 12x_1x_2 + x_2^2$$

* Find The real symmetric matrix C of the quadratic form

$$Q(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + 2x_3^2 + 2x_1x_2 + 6x_2x_3$$

Solⁿ: If $X = [x_1 \ x_2 \ x_3]^T$ we have.

$$Q(x_1, x_2, x_3) = [x_1 \ x_2 \ x_3] \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

આથી આપણે જાણી શકીએ છીએ કે

$$C = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

જે ન આપણે જાણીએ છીએ તે

$$Q(x_1, x_2, x_3) = [x_1 \ x_2 \ x_3] \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= X^T A X$$

§ The real symmetric matrix

$$C = \frac{1}{2} (A + A^T)$$

$$C = \frac{1}{2} \left[\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 6 & 2 \end{bmatrix} \right]$$

$$Q(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 6 \\ 0 & 6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 + 4x_1x_2 + 6x_1x_3 + 2x_2^2 + 12x_2x_3 + 4x_3^2$$

$$Q(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 6 \\ 0 & 6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Q(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 6 \\ 0 & 6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Q. 18 Find the matrix representation of the quadratic form

$$Q(x_1, x_2, x_3) = x_1^2 + 7x_2^2 + 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 6x_2x_3$$

Solⁿ: $X = [x_1 \ x_2 \ x_3]'$

$$Q(x_1, x_2, x_3) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 4 & -2 \\ 0 & 7 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 4 & -2 \\ 0 & 7 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 4 & -2 \\ 0 & 7 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

is the

Answer.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$