CALCULUS II (405)

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UNIT I

* **Taylor's Theorem.** $\sqrt{}$ conti. Condition of schwarz's thm
This is true for more than 2 order partial d.

Let f be a function defined on domain $E \subset \mathbb{R}^2$. If function f possess its continuous partial derivatives of n^{th} order in a nbhd N of point (x, y) in E and $(x+h, y+k) \in N$ then there exist θ such that ~~$0<\theta<1$~~

$$f(x+h, y+k) = f(x,y) + \frac{1}{1!}\left(h f_x(x,y) + k f_y(x,y)\right)$$
$$+ \frac{1}{2!}\left(h^2 f_{xx}(x,y) + 2hk f_{xy}(x,y) + k^2 f_{yy}(x,y)\right)$$
$$+ \dots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x+\theta h, y+\theta k) \quad \text{--- (1)}$$

another form

$x = a$ & $y = b$

$a+h = x'$ & $b+k = y'$

$$f(x', y') = f(a,b) + \frac{1}{1!}\left[(x'-a)\frac{\partial}{\partial x} + (y'-b)\frac{\partial}{\partial y}\right] f(a,b)$$
$$+ \frac{1}{2!}\left[(x'-a)^2\frac{\partial^2}{\partial x^2} + 2(x'-a)(y'-b)\frac{\partial^2}{\partial x \partial y} + (y'-b)^2\frac{\partial^2}{\partial y^2}\right] f(a,b)$$
$$+ \dots + \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n f(a+\theta(x'-a), b+\theta(y'-b)) \quad \text{--- (2)}$$

This is called Taylor's expansion of (x, y) near point (a, b) [or about point (a, b) or in power of $(x'-a)$ & $(y'-b)$].

* Maclaurin's theorem :

Let f be a real function defined on the domain $E \subset \mathbb{R}^2$. If the function f posses its continuous partial derivatives upto n^{th} order in a nbhd $N \subset E$ of the point $(0,0)$ and if $(x,y) \in N$, then there exists $\theta \in (0,1)$ such that

$$f(x,y) = f(0,0) + \frac{1}{1!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0)$$
$$+ \frac{1}{2!}\left(x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x \partial y} + y^2\frac{\partial^2}{\partial y^2}\right) f(0,0)$$
$$+ \cdots\cdots + \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(\theta x, \theta y).$$

* EXAMPLES.

2016 (1) Expand $f(x,y) = x^2y + 3y - 2$ in power of $(x-1)$ and $(y+2)$.

→ Compare $(x-1)$ with $(x-a)$ and $(y+2)$ with $(y-b)$ we get

$a = 1$ and $b = -2$

$$f(x,y) = x^2y + 3y - 2$$
$$= f(1,-2) + \frac{1}{1!}\left[(x-1)f_x(1,-2) + (y+2)f_y(1,-2)\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,-2) + 2(x-1)(y+2)f_{xy}(1,-2) + (y+2)^2 f_{yy}(1,-2)\right]$$
$$+ \frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,-2) + 3(x-1)^2(y+2)f_{xxy}(1,-2) + 3(x-1)(y+2)^2 f_{xyy}(1,-2) + (y+2)^3 f_{yyy}(1,-2)\right]$$
$$+ \cdots$$

$f_x = x^2y \quad f_x = x^2y \quad 2xy$
$f_{xyy} = x^2 \quad f_{xx} = 2xy \quad 2y$
$f_{xxy} = 2x \quad 2$

Here,

$f(x,y) = x^2y+3y-2 \Rightarrow f(1,-2) = 1(-2)+3(-2)-2 = -10$

$f_x(x,y) = 2xy \Rightarrow f_x(1,-2) = 2(1)(-2) = -4$

$f_y(x,y) = x^2+3 \Rightarrow f_y(1,-2) = (1)^2+3 = 4$

$f_{xx}(x,y) = 2y \Rightarrow f_{xx}(1,-2) = 2(-2) = -4$

$f_{xy}(x,y) = 2x \Rightarrow f_{xy}(1,-2) = 2(1) = 2$

$f_{yy}(x,y) = 0 \Rightarrow f_{yy}(1,-2) = 0$

$f_{xxx}(x,y) = 0 \Rightarrow f_{xxx}(1,-2) = 0$

$f_{xxy}(x,y) = 2 \Rightarrow f_{xxy}(1,-2) = 2$

$f_{xyy}(x,y) = 0 \Rightarrow f_{xyy}(1,-2) = 0$

$f_{yyy}(x,y) = 0 \Rightarrow f_{yyy}(1,-2) = 0$

$\therefore f(x,y) = x^2y+3y-2$

$$= -10 + \frac{1}{1!}\left[(x-1)(-4)+(y+2)4\right]$$

$$+ \frac{1}{2!}\left[(x-1)^2(-4)+2(x-1)(y+2)2+(y+2)^2(0)\right]$$

$$+ \frac{1}{3!}\left[(x-1)^3(0)+3(x-1)^2(y+2)2+3(x-1)(y+2)^2(0)+(y+2)^3(0)\right]$$

$$= -10 - 4\left[(x-1)-(y+2)\right] - \frac{4}{2!}\left[(x-1)^2-(x-1)(y+2)\right]$$

$$+ \frac{6}{3!}\left[(x-1)^2(y+2)\right] + \ldots\ldots$$

27] 2016 (2) Expand $f(x,y) = e^{ax}\sin by$ in powers of x & y.

→ $f(x,y) = e^{ax}\sin by$

$$= f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right]$$

$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$+ \ldots\ldots$$

$f(x,y) = e^{ax}\sin by \Rightarrow f(0,0) = 0$
$f_x = a e^{ax}\sin by \Rightarrow f_x(0,0) = 0$
$f_y = b e^{ax}\cos by \Rightarrow f_y(0,0) = b$
$f_{xx} = a^2 e^{ax}\sin by \Rightarrow f_{xx}(0,0) = 0$
$f_{xy} = ab e^{ax}\cos by \Rightarrow f_{xy}(0,0) = ab$
$f_{yy} = -b^2 e^{ax}\sin by \Rightarrow f_{yy}(0,0) = 0$
$f_{xxx} = a^3 e^{ax}\sin by \Rightarrow f_{xxx}(0,0) = 0$
$f_{xxy} = a^2 b e^{ax}\cos by \Rightarrow f_{xxy}(0,0) = a^2 b$
$f_{xyy} = -ab^2 e^{ax}\sin by \Rightarrow f_{xyy}(0,0) = 0$
$f_{yyy} = -b^3 e^{ax}\cos by \Rightarrow f_{yyy}(0,0) = -b^3$

$\therefore f(x,y) = e^{ax}\sin by$

$$= 0 + \frac{1}{1!}\left[x(0) + y(b)\right] + \frac{1}{2!}\left[x^2(0) + 2xy(ab) + y^2(0)\right]$$
$$+ \frac{1}{3!}\left[x^3(0) + 3x^2y(a^2b) + 3xy^2(0) + y^3(-b^3)\right] + \cdots$$

$$= by + xyab + \frac{1}{6}(3x^2y\,a^2b - y^3b^3) + \cdots$$

2013 (3) Expand $f(x,y) = e^{ax}\cos by$ in powers of x and y.

$\rightarrow$ $f(x,y) = e^{ax}\cos by$
$$= f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right]$$
$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$
$$+ \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$
$$+ \cdots$$

$f(x,y) = e^{ax}\cos by \Rightarrow f(0,0) = 1$

$f_x = ae^{ax}\cos by \Rightarrow f_x(0,0) = a$

$f_y = -be^{ax}\sin by \Rightarrow f_y(0,0) = 0$

$f_{xx} = a^2e^{ax}\cos by \Rightarrow f_{xx}(0,0) = a^2$

$f_{xy} = -abe^{ax}\sin by \Rightarrow f_{xy}(0,0) = 0$

$f_{yy} = -b^2e^{ax}\cos by \Rightarrow f_{yy}(0,0) = -b^2$

$f_{xxx} = a^3e^{ax}\cos by \Rightarrow f_{xxx}(0,0) = a^3$

$f_{xxy} = -a^2be^{ax}\sin by \Rightarrow f_{xxy}(0,0) = 0$

$f_{xyy} = -ab^2e^{ax}\cos by \Rightarrow f_{xyy}(0,0) = -ab^2$

$f_{yyy} = +b^3e^{ax}\sin by \Rightarrow f_{yyy}(0,0) = 0$

$\therefore f(x,y) = e^{ax}\cos by$

$$= 1 + \frac{1}{1!}[x(a) + y(0)] + \frac{1}{2!}[x^2(a^2) + 2xy(0) + y^2(-b^2)]$$

$$+ \frac{1}{3!}[x^3(a^3) + 3x^2y\cdot(0) + 3xy^2(-ab^2) + y^3(0)] + \cdots$$

$$f(x,y) = 1 + ax + \frac{1}{2!}[a^2x^2 - y^2b^2] + \frac{1}{3!}[a^3x^3 - 3ab^2xy^2] + \cdots$$

(4) Expand $f(x,y) = \sin x \sin y$ in the power of x and y.

→ $a=0$ and $b=0$

Here $f(x,y) = \sin x \sin y \Rightarrow f(0,0) = 0$

$f_x = \cos x \sin y \Rightarrow f_x(0,0) = 0$

$f_y = \sin x \cos y \Rightarrow f_y(0,0) = 0$

$f_{xx} = -\sin x \sin y \Rightarrow f_{xx}(0,0) = 0$

$f_{xy} = \cos x \cos y \Rightarrow f_{xy}(0,0) = 1$

$f_{yy} = -\sin x \sin y \Rightarrow f_{yy}(0,0) = 0$

$f_{xxx} = -\cos x \sin y \Rightarrow f_{xxx}(0,0) = 0$

$f_{xxy} = -\sin x \cos y \Rightarrow f_{xxy}(0,0) = 0$

$f_{xyy} = -\cos x \sin y \Rightarrow f_{xyy}(0,0) = 0$

$f_{yyy} = -\sin x \cos y \Rightarrow f_{yyy}(0,0) = 0.$

$f_{xxxx} = \sin x \sin y \Rightarrow f_{xxxx}(0,0) = 0$

$f_{xxxy} = -\cos x \cos y \Rightarrow f_{xxxy}(0,0) = -1$

$f_{xxyy} = \sin x \sin y \Rightarrow f_{xxyy}(0,0) = 0$

$f_{xyyy} = -\cos x \cos y \Rightarrow f_{xyyy}(0,0) = \cancel{0} -1$

$f_{yyyy} = +\sin x \sin y \Rightarrow f_{yyyy}(0,0) = 0.$

$\therefore f(x,y) = \sin x \sin y$

$$= f(0,0) + [x f_x(0,0) + y f_y(0,0)]$$
$$+ \frac{1}{2!}[\cancel{(x}\ x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)]$$
$$+ \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)]$$
$$+ \frac{1}{4!}[x^4 f_{xxxx}(0,0) + 4x^3y f_{xxxy}(0,0) + 6x^2y^2 f_{xxyy}(0,0) + 4xy^3 f_{xyyy}(0,0) + y^4 f_{yyyy}(0,0)] + \dots$$

$$= 0 + [x(0) + y(0)] + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(0)]$$
$$+ \frac{1}{3!}[x^3(0) + 3x^2y(0) + 3xy^2(0) + y^3(0)]$$
$$+ \frac{1}{4!}[x^4(0) + 4x^3y(-1) + 6x^2y^2(0) + 4xy^3(-1) + y^4(0)]$$
$$+ \dots$$

$$f(x,y) = xy - \frac{xy}{6}(x^2+y^2) + \dots$$

(5) Expand $f(x,y) = \cancel{x^2y + 3y - 2}\ x^3 + 2x^2y + 3y^2 - 5xy + 3y$ in powers of $(x-1)$, $(y+2)$.

→ Here $a = 1$ and $b = -2$

$f(x,y) = \cancel{x^2y + 3y - 2}\ x^3 + 2x^2y + 3y^2 - 5xy + 3y$

$$= f(1,-2) + [(x-1) f_x(1,-2) + (y+2) f_y(1,-2)]$$
$$+ \frac{1}{2!}[(x-1)^2 f_{xx}(1,-2) + 2(x-1)(y+2) f_{xy}(1,-2) + (y+2)^2 f_{yy}(1,-2)]$$
$$+ \frac{1}{3!}[(x-1)^3 f_{xxx}(1,-2) + 3(x-1)^2(y+2) f_{xxy}(1,-2) + 3(x-1)(y+2)^2 f_{xyy}(1,-2) + (y+2)^3 f_{yyy}(1,-2)]$$
$$+ \dots$$

$f(x,y) = x^3 + 2x^2y + 3y^2 - 5xy + 3y$

$\Rightarrow f(1,-2) = 1 - 4 + 12 + 10 - 6 = 13$

$f_x = 3x^2 + 4xy - 5y \Rightarrow f_x(1,-2) = 3 - 8 + 10 = 5$

$f_y = 2x^2 + 6y - 5x + 3 \Rightarrow f_y(1,-2) = 2 - 12 - 5 + 3 = -12.$

$f_{xx} = 6x + 4y \Rightarrow f_{xx}(1,-2) = 6 - 8 = -2$

$f_{yy} = 6 \Rightarrow f_{yy}(1,-2) = 6$

$f_{xy} = 4x - 5 \Rightarrow f_{xy}(1,-2) = 4 - 5 = -1$

$f_{xxx} = 6 \Rightarrow f_{xxx}(1,-2) = 6$

$f_{xxy} = 4 \Rightarrow f_{xxy}(1,-2) = 4$

$f_{xyy} = 0 \Rightarrow f_{xyy}(1,-2) = 0$

$f_{yyy} = 0 \Rightarrow f_{yyy}(1,-2) = 0.$

$f(x,y) = x^3 + 2x^2y + 3y^2 - 5xy + 3y$

$$= 13 + [(x-1)5 + (y+2)(-12)] + \frac{1}{2!}[(x-1)^2(-2) + 2(x-1)(y+2)(-1) + (y+2)^2(6)] + \frac{1}{3!}[(x-1)^3(6) + 3(x-1)^2(y+2)4$$
$$+ 3(x-1)(y+2)^2(0) + (y+2)^3(0)] + \dots$$

$$= 13 + 5(x-1) - 12(y+2) - (x-1)^2 - (x-1)(y+2) + 3(y+2)^2$$
$$+ (x-1)^3 + 2(x-1)^2(y+2).$$

(6) Prove that :

2015

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}\left[(x-1)^2 - (y-1)^2\right] + \ldots$$

→ Here $a=1$ and $b=1$

$f(x,y) = \tan^{-1}\frac{y}{x}$

$$f(x,y) = f(1,1) + \left[(x-1) f_x(1,1) + (y-1) f_y(1,1)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1) f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right]$$

$$+ \frac{1}{3!}\left[(x-1)^3 f_{xxx}(1,1) + 3(x-1)^2(y-1) f_{xxy}(1,1) + 3(x-1)(y-1)^2 f_{xyy}(1,1) + (y-1)^3 f_{yyy}(1,1)\right]$$

$+ \ldots$

$f(x,y) = \tan^{-1}\frac{y}{x} \Rightarrow f(1,1) = \tan^{-1} 1 = \frac{\pi}{4}$

$f_x = \dfrac{1}{1+\frac{y^2}{x^2}}\left(\dfrac{-y}{x^2}\right) = \dfrac{-y}{x^2+y^2} \Rightarrow f_x(1,1) = \dfrac{-1}{1+1} = -\dfrac{1}{2}$

$f_y = \dfrac{1}{1+\frac{y^2}{x^2}}\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2+y^2} \Rightarrow f_y(1,1) = \dfrac{1}{2}$

$f_{xx} = \dfrac{+2xy}{(x^2+y^2)^2} \Rightarrow f_{xx}(1,1) = \dfrac{+2}{4} = \dfrac{1}{2}$

$f_{yy} = \dfrac{-2xy}{(x^2+y^2)^2} \Rightarrow f_{yy}(1,1) = \dfrac{-2}{4} = -\dfrac{1}{2}$

$f_{xy} = \dfrac{(x^2+y^2)(-1) - (-y)2y}{(x^2+y^2)^2} \Rightarrow f_{xy}(1,1) = 0$

$\therefore \tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{\pi}{4} + \left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right]$

$+ \dfrac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)\,0 + (y-1)^2\left(-\frac{1}{2}\right)\right]$

$\tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{\pi}{4} - \dfrac{1}{2}(x-1) + \dfrac{1}{2}(y-1) + \dfrac{1}{4}\left[(x-1)^2 - (y-1)^2\right] + \ldots$

Hence proved.

Extra Ex. $f(x,y) = x^3 + xy^2 + x^2y + y^3$, using Taylor's theorem expressing $f(3,1) - f(2,-1)$ upto second partial derivative and prove that $\theta = 3/8$.

(7) Prove the following :

If $f(x,y) = 2x^2 - 3xy + y^2$, using Taylor's theorem, expressing $f(2,1) - f(-1,2)$ upto first partial derivatives, prove that $\theta = \frac{1}{2}$.

→ $f(x,y) = 2x^2 - 3xy + y^2$

$f(2,1) = 2(4) - 3(2)(1) + (1)^2 = 8 - 6 + 1 = 3$

$f(-1,2) = 2(1) - 3(-1)(2) + (4) = 2 + 6 + 4 = 12$

$f_x(x,y) = 4x - 3y \Rightarrow f_x(x+\theta h, y+\theta k) = 4(x+\theta h) - 3(y+\theta k)$

$f_y(x,y) = -3x + 2y \Rightarrow f_y(x+\theta h, y+\theta k) = -3(x+\theta h) + 2(y+\theta k)$

$(2,1) = (-1+3, 2-1)$; $x+h$, $y+k$

Let $x=-1$, $y=2$, $h=3$ & $k=-1$

We know that

$f(x+h, y+k) = f(x,y) + h f_x(x+\theta h, y+\theta k) + k f_y(x+\theta h, y+\theta k)$

$\therefore f(2,1) = f(-1,2) + 3 f_x(-1+3\theta, 2-\theta) + (-1) f_y(-1+3\theta, 2-\theta)$

$\therefore 3 = 12 + 3\left[4(-1+3\theta) - 3(2-\theta)\right] - \left[-3(-1+3\theta) + 2(2-\theta)\right]$

$\therefore 3 = 12 + 3\left[-4 + 12\theta - 6 + 3\theta\right] - \left[3 - 9\theta + 4 - 2\theta\right]$

$\therefore 3 = 12 + 3\left[-10 + 15\theta\right] - \left[7 - 11\theta\right]$

$\therefore 3 = 12 - 30 + 45\theta - 7 + 11\theta$

$\therefore 3 - 12 + 7 + 30 = 56\theta$

$\therefore 28 = 56\theta$

$\therefore \theta = \dfrac{28}{56}$

$\therefore \boxed{\theta = \dfrac{1}{2}}$

(8) Expand $f(x,y,z) = xyz$ in powers of $x+1, y+1, z+2$.

→ Here $a=-1$, $b=-1$ and $c=-2$.

$$f(x,y,z) = xyz$$
$$= f(a,b,c) + \sum_{n=1}^{n-1} \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} + (z-c)\frac{\partial}{\partial z}\right]^n f(a,b,c) + \dots$$
$$= f(a,b,c) + \frac{1}{1!}\left[(x-a)f_x(a,b,c) + (y-b)f_y(a,b,c) + (z-c)f_z(a,b,c)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b,c) + (y-b)^2 f_{yy}(a,b,c) + (z-b)^2 f_{zz}(a,b,c) + 2(x-a)(y-b)f_{xy}(a,b,c) + 2(y-b)(z-c)f_{yz}(a,b,c) + 2(z-c)(x-a)f_{zx}(a,b,c)\right] + \dots$$

$f(x,y,z) = xyz \Rightarrow f(-1,-1,-2) = -2$

$f_x = yz \Rightarrow f_x(-1,-1,-2) = 2$

$f_y = xz \Rightarrow f_y(-1,-1,-2) = 2$

$f_z = xy \Rightarrow f_z(-1,-1,-2) = 1$

$f_{xx} = 0 \Rightarrow f_{xx}(-1,-1,-2) = 0$

$f_{yy} = 0 \Rightarrow f_{yy}(-1,-1,-2) = 0$

$f_{zz} = 0 \Rightarrow f_{zz}(-1,-1,-2) = 0$

$f_{xy} = z \Rightarrow f_{xy}(-1,-1,-2) = -2$

$f_{yz} = x \Rightarrow f_{yz}(-1,-1,-2) = -1$

$f_{zx} = y \Rightarrow f_{zx}(-1,-1,-2) = -1$

$$\therefore f(x,y,z) = -2 + \left[(x+1)2 + (y+1)2 + (z+2)1\right]$$
$$+ \frac{1}{2!}\left[(x+1)^2(0) + (y+1)^2(0) + (z+2)^2(0) + 2(x+1)(y+1)(-2) + 2(y+1)(z+2)(-1) + 2(z+2)(x+1)(-1)\right] + \dots$$

$$\therefore xyz = -2 + 2(x+1) + 2(y+1) + (z+2) + \left[2(x+1)(y+1) - (y+1)(z+2) - (z+2)(x+1)\right] \dots$$

UNIT II

✱ Extreme values of real functions of two variables :

[1] Maximum and minimum values of $f(x,y)$.

Let f be a real valued function defined on an open domain $E \subset \mathbb{R}^2$. If N be the nbhd of point $(a,b) \in E$ and $(a+h, b+k) \in N \cap E$, then

(i) $f(a,b)$ is said to be the local maximum value of $f(x,y)$ at point (a,b) if $f(a,b) \geq f(a+h, b+k)$, $\forall (a+h, b+k) \in N \cap E$.

(ii) $f(a,b)$ is said to be the ~~locat~~ local minimum value of $f(x,y)$ at point (a,b) if $f(a,b) \leq f(a+h, b+k)$, $\forall (a+h, b+k) \in N \cap E$.

[2] Extreme values of function $f(x,y)$.

Let f be a real valued function defined on an open domain $E \subset \mathbb{R}^2$, a local maximum or a local minimum value of $f(x,y)$ at point $(a,b) \in E$ is said to be the local extreme value of $f(x,y)$ at point (a,b).

[3] Absolute (or global) extreme value of $f(x,y)$.

Let f be a real valued function defined on an open domain $E \subset \mathbb{R}^2$ and $(a,b) \in E$ and

(i) If $f(a,b) \geq f(x,y)$, $\forall (x,y) \in E$, then $f(a,b)$ is said to be absolute maximum value of $f(x,y)$ at point (a,b).

(ii) If $f(a,b) \leq f(x,y)$, $\forall (x,y) \in E$, the $f(a,b)$ is said to be absolute minimum value of $f(x,y)$ at point (a,b).

(iii) If $f(a,b)$ is either absolute maximum or absolute minimum value, then $f(a,b)$ is said to be absolute extreme value of $f(x,y)$ at point (a,b).

Theorem: The necessary conditions that a real valued funt f, defined on an open domain $E \subset \mathbb{R}^2$ and is differentiable at point $(a,b) \in E$, has an extreme value at (a,b) are $f_x(a,b) = 0$, $f_y(a,b) = 0$.

Proof: Let us suppose that $f(a,b)$ is the maximum value of $f(x,y)$ at point $(a,b) \in E$, therefore there exists nbhd N of point (a,b) such that

$\forall (x,y) \in N \cap E \Rightarrow f(a,b) \geq f(x,y)$

$\Rightarrow f(a,b) \geq f(a+h,b)$, $(a+h,b) \in N \cap E$.

$$\frac{f(a+h,b) - f(a,b)}{h} \leq 0 \quad \text{if } h \in \mathbb{R}^+$$

and $$\frac{f(a+h,b) - f(a,b)}{h} \geq 0 \quad \text{if } h \in \mathbb{R}^-$$

$\therefore \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \leq 0$ and

$\lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \geq 0$.

$(h \to 0^+)$ $\quad$ $(h \to 0^-)$

$\therefore R f_x(a,b) \leq 0$ and $L f_x(a,b) \geq 0$. —— (1)

Since f is differentiable at point (a,b).

[$\because$ limit of both sides exist and equal].

$\therefore L f_x(a,b) = R f_x(a,b) = f_x(a,b)$ —— (2)

$\therefore$ From (1) & (2)

$f_x(a,b) = 0$.

Similarly, we can prove that $f_y(a,b) = 0$.

$\therefore$ The necessary condition that the function $f(x,y)$ has a maximum value at point (a,b) is
$f_x(a,b) = 0$, $f_y(a,b) = 0$.

Similarly, the necessary conditions that the function $f(x,y)$ has minimum value at point (a,b) are
$f_x(a,b) = 0$, $f_y(a,b) = 0$.

NOTE: The converse of these theorem is not true. If for function $f(x,y)$, $f_x(a,b) = 0$, $f_y(a,b) = 0$ for $(a,b) \in D_f$, then $f(a,b)$ may or may not be maximum or minimum value of (x,y) at point (a,b).

Illustration:

let $f(x,y) = xy$, $0 < x^2+y^2 < 1$.

$\therefore f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$

& $f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$

$\therefore f_x(0,0) = 0$ and $f_y(0,0) = 0$

Now

$f(x,y) = xy > 0$ if (x,y) lies 1^{st} or 3^{rd} quandrant (अर्थात)

$= xy < 0$ if (x,y) lies 2^{nd} or 4^{th} quandrant

and $f(0,0) = 0$

$\therefore \forall (x,y) \in N$ (the nbhd of $(0,0)$), $f(x,y) \nless f(0,0)$ & $f(x,y) \ngtr f(0,0)$

$\therefore f(0,0)$ is neither maximum nor minimum value of $f(x,y)$ at point $(0,0)$.

Theorem 2: Sufficient conditions:

Let a function f possess continuous second order partial derivatives at point (a,b) in open domain $E \subset \mathbb{R}^2$ and $f_x(a,b) = f_y(a,b) = 0$. If $f_{xx}(a,b) = r$, $f_{xy}(a,b) = s$ and $f_{yy}(a,b) = t$, then if

(1) $rt - s^2 > 0$ and $r < 0$, then the function f has a maximum value at point (a,b).

(2) $rt - s^2 > 0$ and $r > 0$, then the function f has a minimum value at point (a,b).

(3) $rt - s^2 < 0$, then the function f has neither maximum nor minimum value at point (a,b).

(4) $rt - s^2 = 0$, then the test is inconclusive. (Fail).

Proof: By Taylor's theorem,

$$f(a+h, b+k) = f(a,b) + \sum_{r=1}^{n-1} \frac{1}{r!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^r f(a,b)$$
$$+ \frac{1}{n!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^n f(a+\theta h, b+\theta k).$$

$$\therefore f(a+h, b+k) - f(a,b) = [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2}[rh^2 + 2shk + tk^2] + \rho$$

where ρ is the third and more degree in h and k such that $0 < h^2+k^2 < \rho$.

But $f_x(a,b) = 0$, $f_y(a,b) = 0$

$$\therefore f(a+h, b+k) - f(a,b) = \frac{1}{2}(rh^2 + 2shk + tk^2) + \rho \quad \text{---(1)}$$

Hence for determining the sign of L.H.S. of (1), ρ can be neglected, hence sign is dependent of G

where $G = rh^2 + 2shk + tk^2$

$$= \frac{1}{r}\left[(rh+sk)^2 + (rt - s^2)k^2\right]$$

(1) If $r < 0$, $rt - s^2 > 0$, $h \neq 0$, $k \neq 0 \Rightarrow G < 0$

$\Rightarrow f(a+h, b+k) - f(a,b) < 0$.

In this case function f has maximum value at point (a,b).

(2) If $r > 0$, $rt - s^2 > 0$, $h \neq 0$, $k \neq 0 \Rightarrow G > 0$

$\Rightarrow f(a+h, b+k) - f(a,b) > 0$.

In this case function f has minimum value at point (a,b).

(3) If $rt - s^2 < 0 \Rightarrow G$ and hence $f(a+h, b+k) - f(a,b)$ may be positive, negative or zero.
In this case f has neither maximum nor minimum value at point (a,b). In this case if $f(a+h, b) - f(a,b)$ and $f\left(a - \frac{ks}{r}, b+k\right) - f(a,b)$ have opposite signs, such a point is sometimes called *saddle point.

*Result does not obtain directly by test

(4) If $rt - s^2 = 0$, then $f(a+h, b+k) - f(a,b) = \frac{1}{r}(rh+sk)^2$
In this case the test is * inconclusive (ambiguous or doubtful) and further investigation is needed to determine whether $f(x,y)$ is a maximum or minimum at $(0,0)$ or not.

* A point (a,b) of E is called a saddle point if every nbhd N of (a,b) contains points (x,y) such that $f(x,y) < f(a,b)$ and other points (x,y) such that $f(x,y) > f(a,b)$.

* Find the extreme values of the followings :

(1) $f(x,y) = x^3 + y^3 - 3x - 12y + 5$.

$\Rightarrow f_x = 3x^2 - 3$ & $f_y = 3y^2 - 12$

Let $f_x = 0$ & $f_y = 0$

$\Rightarrow 3x^2 - 3 = 0$ & $3y^2 - 12 = 0$

$\Rightarrow x^2 - 1 = 0$ $\quad y^2 - 4 = 0$

$\Rightarrow x^2 = 1$ $\quad y^2 = 4$

$\Rightarrow x = \pm 1$ $\quad y = \pm 2$

$f_{xx} = 6x$

$f_{xy} = 0$

$f_{yy} = 6y$

(a) Let $(x,y) = (1,2)$

$\therefore r = 6, \; s = 0, \; t = 12$

$\therefore rt - s^2 = 72 > 0$, since $r = 6 > 0$

$\therefore (x,y) = (1,2)$ is minima.

$\therefore$ Minimum value is

$f(1,2) = 1 + 8 - 3 - 24 + 5 = -13$.

(b) Let $(x,y) = (-1,2)$

$\therefore r = -6, \; s = 0, \; t = 12$

$\therefore rt - s^2 = -72 < 0$.

No minima, No maxima.

(c) Let $(x,y) = (1,-2)$

$\therefore r = 6, \; s = 0, \; t = -12$

$\therefore rt - s^2 = -72 < 0$

No minima, No maxima.

(d) Let $(x,y) = (-1,-2)$

$r = -6, \; s = 0, \; t = -12$

$\therefore rt - s^2 = 72 > 0$ Since $r = -6 < 0$

$\therefore (x,y) = (-1,-2)$ is maxima

$\therefore$ maximum value is

$f(-1,-2) = -1 - 8 + 3 + 24 + 5 = 23$.

(2) $f(x,y) = x^3 + y^3 - 3axy$

$\Rightarrow f_x = 3x^2 - 3ay$ & $f_y = 3y^2 - 3ax$

Let $f_x = 0$ & $f_y = 0$

$\Rightarrow 3x^2 - 3ay = 0$ & $3y^2 - 3ax = 0$

$\Rightarrow 3x^2 = 3ay$ $\quad 3y^2 = 3ax$

$\Rightarrow x^2 = ay$ —(1) $\quad y^2 = ax$ —(2)

$\Rightarrow y = x^2/a$ put in eq^n (2) we get

$\frac{x^4}{a^2} = ax \Rightarrow x^4 = a^3 x$

$\Rightarrow x^4 - a^3 x = 0 \Rightarrow x(x^3 - a^3) = 0$

$\Rightarrow \boxed{x = 0}$ or $\boxed{x = a}$

If $x = 0 \Rightarrow \boxed{y = 0}$ $\quad \left(\because y = \frac{x^2}{a}\right)$

$x = a \Rightarrow \boxed{y = a}$

$f_{xx} = 6x$, $f_{xy} = -3a$ & $f_{yy} = 6y$

(a) Let $(x,y) = (0,0)$

$\therefore r = 0, \; s = -3a$ & $t = 0$

$\therefore rt - s^2 = -9a^2 < 0$

No maxima, no minima.

(b) Let $(x,y) = (a,a)$

$\therefore r = 6a,\ s = -3a$ & $t = 6a$

$\therefore rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

Since $r = 6a$.

If $a > 0$, minimum value at (a,a)

$\therefore$ minimum value is

$f(a,a) = a^3 + a^3 - 3a^3$

$= -a^3$

If $a < 0$, maximum value at (a,a).

$\therefore$ Maximum value is

$f(a,a) = -a^3$.

2016 (3) $f(x,y) = 2(x-y)^2 - x^4 - y^4$

$\rightarrow f_x = 4(x-y) - 4x^3$ & $f_y = -4(x-y) - 4y^3$

Let $f_x = 0$ & $f_y = 0$

$\Rightarrow 4(x-y) - 4x^3 = 0$ & $-4(x-y) - 4y^3 = 0$

$\Rightarrow 4(x-y) = 4x^3$ $-4(x-y) = 4y^3$

$\Rightarrow x - y = x^3$ — (1) $-(x-y) = y^3$ — (2)

$\Rightarrow \cancel{x^3 - x = y}$ $\cancel{y^3 - y = -x}$

$\Rightarrow -x^3 = y^3$

$\Rightarrow \boxed{x=0}\ \boxed{y=0}$ & $-x = y \Rightarrow x + y = 0$

$\Rightarrow \boxed{x = -y}$ — (3)

From (1) & (3)

$x = x^3 \Rightarrow 2x - x^3 = 0 \Rightarrow x(2 - x^2) = 0$

$\Rightarrow x = 0,\ \boxed{x = \pm\sqrt{2}}$ hence $\boxed{y = \mp\sqrt{2}}$

~~Let $(x,y) = (0,0)$~~

$=$

$f_{xx} = 4 - 12x^2,\ f_{xy} = -4$ & $f_{yy} = 4 - 12y^2$

(a) Let $(x,y) = (0,0)$.

$r = 4,\ s = -4,\ t = 4$

$\therefore rt - s^2 = 16 - 16 = 0$

$\therefore$ No.

(b) Let $(x,y) = (\sqrt{2}, -\sqrt{2})$

$r = 4 - 24 = -20,\ s = -4$ & $t = 4 - 24 = -20$

& $rt - s^2 = 400 - 16 = 384 > 0$

Since $r = -20 < 0$

$\therefore (x,y) = (\sqrt{2}, -\sqrt{2})$ is maxima.

$\therefore$ Maximum value is

$f(\sqrt{2}, -\sqrt{2}) = 2(\sqrt{2} + \sqrt{2})^2 - (\sqrt{2})^4 - (-\sqrt{2})^4$

$= 2 \times 4 \times 2 - 4 - 4$

$= 8$

(c) Let $(x,y) = (-\sqrt{2}, \sqrt{2})$

$r = 4 - 24 = -20,\ s = -4$ & $t = 4 - 24 = -20$

$rt - s^2 = 400 - 16 = 384 > 0$

Since $r = -20 < 0$

$\therefore (x,y) = (-\sqrt{2}, \sqrt{2})$ is maxima.

$\therefore$ Maximum value is

$f(-\sqrt{2}, \sqrt{2}) = 2(-\sqrt{2} + \sqrt{2})^2 - (-\sqrt{2})^4 - (\sqrt{2})^4$

$= 2 \times 4 \times 2 - 4 - 4$

$= 8$

2015 (4) $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

$\rightarrow f_x = 3x^2 - 3$ & $f_y = 3y^2 - 12$

$f_{xx} = 6x,\ f_{xy} = 0,\ f_{yy} = 6y$

Let $f_x = 0$ and $f_y = 0$

$\Rightarrow 3x^2 - 3 = 0$ & $3y^2 - 12 = 0$

$\Rightarrow x^2 - 1 = 0$ $y^2 - 4 = 0$

$\Rightarrow x^2 = 1$ & $y^2 = 4$

$\Rightarrow x = \pm 1$ $\quad y = \pm 2$

(a) Let $(x, y) = (1, 2)$

$r = 6$, $s = 0$, $t = 12$

$rt - s^2 = 72 > 0$

Since $r = 6 > 0$

$\therefore (x, y) = (1, 2)$ is minima.

$\therefore$ minimum value is

$f(1,2) = 1 + 8 - 3 - 24 + 20 = 2$

(b) Let $(x, y) = (-1, 2)$

$r = -6$, $s = 0$, $t = 12$

$rt - s^2 = -72 < 0$

No minima, No maxima

(c) Let $(x, y) = (1, -2)$

$r = 6$, $s = 0$, $t = -12$

$rt - s^2 = -72 < 0$

No minima, No maxima.

(d) Let $(x, y) = (-1, -2)$

$r = -6$, $s = 0$, $t = -12$

$rt - s^2 = 72 > 0$

Since $r = -6 < 0$

$\therefore (x, y) = (-1, -2)$ is maxima

$\therefore$ Maximum value is

$f(-1, -2) = -1 - 8 + 3 + 24 + 20 = 38$

(6) $\cancel{f(x, y, z)} = x^2 + y^2 + z^2$, ~~where~~ $\cancel{6x + 24y + 8z = 7}$.

$\rightarrow \cancel{f_x = 2x}$

(5) $f(x, y) = xy + a^3 \left[\frac{1}{x} + \frac{1}{y}\right]$.

$\rightarrow f_x = y - \frac{a^3}{x^2}$ & $f_y = x - \frac{a^3}{y^2}$

$f_{xx} = \frac{2a^3}{x^3}$, $f_{xy} = 1$, $f_{yy} = \frac{2a^3}{y^3}$

Let $f_x = 0$ & $f_y = 0$

$\Rightarrow y - \frac{a^3}{x^2} = 0$ — (1) & $x - \frac{a^3}{y^2} = 0$ — (2)

$\Rightarrow y = \frac{a^3}{x^2}$ put in (2) we get

$x - \frac{a^3}{\left(\frac{a^3}{x^2}\right)^2} = 0$

$\Rightarrow x - \frac{x^4}{a^3} = 0$

$\Rightarrow a^3 x - x^4 = 0$

$\Rightarrow x(a^3 - x^3) = 0$

$\Rightarrow \boxed{x = 0}$ & $\boxed{x = a}$

If $x = 0 \Rightarrow y$ does not exists.

If $x = a \Rightarrow \boxed{y = a}$

Let $(x, y) = (a, a)$.

$r = \cancel{a}\ \frac{a^3}{\cancel{a^2}}\ 2\cancel{a}$, $s = 1$ & $t = 2$

$\therefore rt - s^2 = 4 - 1 = 3 > 0$

Since $r = 2 > 0$

$\therefore (x, y) = (a, a)$ is minima.

$\therefore$ Minimum value is

$f(a, a) = a^2 + a^3 \left[\frac{1}{a} + \frac{1}{a}\right] = a^2 + 2a^2 = 3a^2$.

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(6) $f(x,y) = \sin^2 x + \sin^2 y$; $x+y = a$ and $a \neq (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$.

$f_x = \sin 2x$ & $f_y = \sin 2y$

$f_{xx} = 2\cos 2x$, $f_{xy} = 0$, $f_{yy} = 2\cos 2y$

let $f_x = 0$ & $f_y = 0$

$\Rightarrow \sin 2x = 0$ & $\sin 2y = 0$

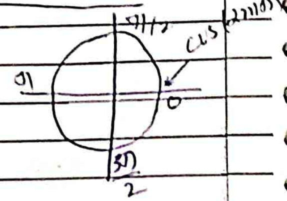
$\Rightarrow 2x = k\pi ; k \in \mathbb{Z}$ $\quad 2y = k\pi ; k \in \mathbb{Z}$

$\Rightarrow x = \frac{k\pi}{2} ; k \in \mathbb{Z}$ $\quad y = \frac{k\pi}{2} ; k \in \mathbb{Z}$

$\therefore (x,y) = \left(\frac{k\pi}{2}, \frac{k\pi}{2}\right)$ for $k \in \mathbb{Z}$

(a) Let $(x,y) = \left(\frac{k\pi}{2}, \frac{k\pi}{2}\right)$ and let $k = 2m, m \in \mathbb{Z}$

$\therefore r = 2\cos\left(2 \cdot \frac{2m\pi}{2}\right) = 2\cos(2m\pi) = 2$



$s = 0$ & $t = 2$

$\therefore rt - s^2 = 4 > 0$ & $r > 0$

$\therefore (x,y) = \left(\frac{k\pi}{2}, \frac{k\pi}{2}\right)$ where $k = 2m \ (m \in \mathbb{Z})$ is minima and gives us minimum value.

$\therefore f\left(\frac{k\pi}{2}, \frac{k\pi}{2}\right) = \sin^2\frac{k\pi}{2} + \sin^2\frac{k\pi}{2}$

$= \sin^2(m\pi) + \sin^2(m\pi)$

$= 0$

$\therefore$ Minimum value is 0 at point $\left(\frac{k\pi}{2}, \frac{k\pi}{2}\right)$; $k = 2m$.

(b) Let $(x,y) = \left(\frac{k\pi}{2}, \frac{k\pi}{2}\right)$ and let $k = (2m+1)$; $m \in \mathbb{Z}$

$\therefore r = 2\cos((2m+1)\pi) = -2$, $s = 0$ & $t = -2$

$\cos(2m+1)\pi$

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$\therefore rt - s^2 = 4 > 0$ & $r = -2 < 0$

$\therefore (x,y) = \left(\frac{k\pi}{2}, \frac{k\pi}{2}\right)$; $k = 2m+1$; $m \in \mathbb{Z}$ gives us maximum value.

$\therefore f\left(\frac{k\pi}{2}, \frac{k\pi}{2}\right) = \sin^2\frac{(2m+1)\pi}{2} + \sin^2\frac{(2m+1)\pi}{2}$

$= 2$

$\therefore$ Maximum value is 2.

(7) $f(x,y) = \sin x + \sin y + \cos(x+y)$.

$\rightarrow f_x = \cos x - \sin(x+y)$ & $f_y = \cos y - \sin(x+y)$

Let $f_x = 0$ & $f_y = 0$

$\Rightarrow \cos x - \sin(x+y) = 0$ & $\cos y - \sin(x+y) = 0$

$\Rightarrow \cos x = \sin(x+y)$ —(1) $\quad \cos y = \sin(x+y)$ —(2)

$\Rightarrow \cos x = \cos y \not\Rightarrow$ cos in 1 quadrant

If we take $(2k+1)\pi$

$\Rightarrow x = y + 2k\pi$; $k \in \mathbb{Z}$ —(3)

From (1) & (3) we get

we get same answer

$\cos(y + 2k\pi) = \sin(2y + 2k\pi)$

first quadrant so Both cos and sin are positive

$\Rightarrow \cos y = \sin 2y$

Will it take $(2k+1)\pi$

$\Rightarrow \cos y = 2\sin y \cos y$

$\Rightarrow \cos y - 2\sin y \cos y = 0$

$\Rightarrow \cos y(1 - 2\sin y) = 0$

y will remain

$\Rightarrow \cos y = 0$ or $1 - 2\sin y = 0$

$\Rightarrow y = (2k+1)\frac{\pi}{2}$; $k \in \mathbb{Z}$ $\quad \therefore \sin y = \frac{1}{2}$

$\therefore y = \frac{\pi}{6} + 2k\pi$; $k \in \mathbb{Z}$

Since $x = y + 2k\pi$

For $\boxed{y = (2k+1)\frac{\pi}{2}}$; $k \in \mathbb{Z}$ $\Rightarrow \boxed{x = (2k+1)\frac{\pi}{2} + 2k\pi}$

For $\boxed{y = \frac{\pi}{6} + 2k\pi}$; $k \in \mathbb{Z}$ $\Rightarrow x = \frac{\pi}{6} + 4k\pi = \frac{\pi + 12k\pi}{6}$

$\Rightarrow \boxed{x = \frac{\pi}{6} + 2k\pi}$

(+) ~~let $(x,y) = \left((2k+1)\frac{\pi}{2} + 2k\pi,\ (2k+1)\frac{\pi}{2}\right)$ and let $k = 2m$~~

$f_{xx} = -\sin x - \cos(x+y)$
$f_{xy} = -\cos(x+y)$
$f_{yy} = -\sin y - \cos(x+y)$.

(i) Let $(x,y) = \left((2k+1)\frac{\pi}{2} + 2k\pi,\ (2k+1)\frac{\pi}{2}\right)$ and Let $k = 2m;\ m \in \mathbb{Z}$

$\therefore r = -\sin\left((4m+1)\frac{\pi}{2} + 4m\pi\right) - \left(\cos\ (4m+1)\frac{\pi}{2} + 4m\pi + (4m+1)\frac{\pi}{2}\right)$

$= -\sin\left[(4m+1)\frac{\pi}{2}\right] - \cos\left[(4m+1)\pi\right]$

$= -1 - (-1)$

$\therefore r = 0$

$1 = \sin(4m+1)\frac{\pi}{2}$

$-1 = \cos(4m+1)\pi$

$s = -\cos\left[(4m+1)\frac{\pi}{2} + 4m\pi + (4m+1)\frac{\pi}{2}\right]$
$= -\cos\left[(4m+1)\pi\right]$
$= -(-1)$
$= 1$

$-1 = \cos(4m+1)\pi$

$t = -\sin\left[(4m+1)\frac{\pi}{2}\right] - \cos\left[(4m+1)\frac{\pi}{2} + 4m\pi + (4m+1)\frac{\pi}{2}\right]$

$= -1 - (-1)$
$= 0$

$\therefore rt - s^2 = -1 < 0$

$\therefore$ No minima, No maxima.

(ii) Let $(x,y) = \left[(2k+1)\frac{\pi}{2} + 2k\pi,\ (2k+1)\frac{\pi}{2}\right]$ and

Let $k = (2m+1)$; $m \in \mathbb{Z}$

$\therefore r = -\sin\left[(2(2m-1)+1)\frac{\pi}{2} + 2(2m+1)\pi\right]$

$- \cos\left[(2(2m+1)+1)\frac{\pi}{2} + 2(2m+1)\pi + (2(2m+1)+1)\frac{\pi}{2}\right]$

$= -\sin\left[(4m-1)\frac{\pi}{2} + (4m+2)\pi\right]$

$- \cos\left[(4m-1)\pi + (4m-2)\pi\right]$

$= -\sin\left[(4m-1)\frac{\pi}{2}\right] - \cos\left[(4m-1)\pi\right]$

$-1 = \sin(4m-1)\frac{\pi}{2}$

$\cos(4m-1)\pi = -1$

$= -(-1) - (-1)$

$= 1 + 1$

$\therefore r = 2$

if we take $k = 2m+1$ then we get $(4m+3)$

$s = -\cos\left[(2(2m-1)+1)\frac{\pi}{2} + 2(2m-1)\pi + (2(2m-1)+1)\frac{\pi}{2}\right]$

$= -\cos\left[(4m-1)\pi + (4m-2)\pi\right]$

$= -\cos\left[(4m-1)\pi\right]$.

$= -(-1)$

$= 1$

$t = 2$

$\therefore rt - s^2 = 4 - 1 = 3 > 0,\ r > 0$

$\therefore (x,y) = \left[(2k+1)\frac{\pi}{2} + 2k\pi,\ (2k+1)\frac{\pi}{2}\right],\ k = (2m-1)$
$m \in \mathbb{Z}$

has mi is minima and gives us minimum value.

$\therefore f\left[(2k+1)\frac{\pi}{2} + 2k\pi,\ (2k+1)\frac{\pi}{2}\right] = \sin\left[(2k+1)\frac{\pi}{2} + 2k\pi\right] + \sin(2k+1)\frac{\pi}{2}$

$+ \cos\left[(2k+1)\frac{\pi}{2} + 2k\pi + (2k+1)\frac{\pi}{2}\right]$

$= \sin(2k+1)\frac{\pi}{2} + \sin(2k+1)\frac{\pi}{2}$
$+ \cos(2k+1)\pi$

$= \cancel{\sin \quad \cos}$

$= 2\sin(2(2m-1)+1)\pi/2 + \cos(2(2m-1)+1)\pi$

$= 2\sin(4m-1)\pi/2 + \cos(4m-1)\pi$

$= 2(-1) + (-1)$

$= -2 - 1$

$= -3$

$\therefore$ Minimum value is -3 at $\left[(2k+1)\frac{\pi}{2} + 2k\pi, (2k+1)\frac{\pi}{2}\right]$
$k = 2m-1 \; ; m \in \mathbb{Z}$

(iii) Let $(x, y) = \left(\frac{\pi}{6} + 2k\pi, \frac{\pi}{6} + 2k\pi\right)$

$\therefore r = -\sin\left(\frac{\pi}{6} + 2k\pi\right) - \cos\left(\frac{\pi}{6} + 2k\pi + \frac{\pi}{6} + 2k\pi\right)$

$= -\sin\frac{\pi}{6} - \cos\frac{\pi}{3}$

$= -\frac{1}{2} - \frac{1}{2}$

$= -1$

$s = -\cos\left(\frac{\pi}{6} + 2k\pi + \frac{\pi}{6} + 2k\pi\right)$

$= -\cos\pi/3$

$= -\frac{1}{2}$

$t = -1$

$\therefore rt - s^2 = 1 - \frac{1}{4} = \frac{3}{4} > 0$ & $r = -1 < 0$

$\therefore (x, y) = \left(\frac{\pi}{6} + 2k\pi, \frac{\pi}{6} + 2k\pi\right)$; $k \in \mathbb{Z}$ & is maxima and gives us maximum value

$f\left(\pi/6 + 2k\pi, \pi/6 + 2k\pi\right) = \sin\left(\frac{\pi}{6} + 2k\pi\right) + \sin\left(\frac{\pi}{6} + 2k\pi\right)$
$+ \cos\left(\frac{\pi}{6} + 2k\pi + \frac{\pi}{6} + 2k\pi\right)$

$= 2\sin\frac{\pi}{6} + \cos\frac{\pi}{3}$

$= 2\left(\frac{1}{2}\right) + \frac{1}{2}$

$= 1 + \frac{1}{2}$

$= \frac{3}{2}$

$\therefore$ Maximum value is $\frac{3}{2}$ at $\left(\frac{\pi}{6} + 2k\pi, \frac{\pi}{6} + 2k\pi\right)$
$k \in \mathbb{Z}$.

(8) Find three positive numbers whose sum is 24 and their product is maximum.

→ Let us suppose that x, y and z are the required numbers.

$f(x, y) = xyz$ & $x + y + z = 24$

$\therefore z = 24 - (x + y)$

$\therefore f(x, y) = xy\left[24 - (x+y)\right]$

$= 24xy - x^2y - xy^2$

$\therefore f_x = 24y - 2xy - y^2$ & $f_y = 24x - x^2 - 2xy$

Let $f_x = 0$ & $f_y = 0$

$\therefore 24y - 2xy - y^2 = 0$ —(1) & $24x - x^2 - 2xy = 0$ —(2)

$\therefore 2xy = 24y - y^2$ $2xy = 24x - x^2$

$\therefore 24y - y^2 = 24x - x^2$

$\therefore 24y - 24x = -x^2 + y^2$

$\therefore 24(y - x) = (y - x)(y + x)$

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$\therefore 24(y-x) - (y-x)(y+x) = 0$

$\therefore (y-x)[24 - y - x] = 0$

$\therefore y - x = 0$ or $24 - y - x = 0$

$\therefore y = x$ $\qquad \therefore x + y = 24$

If $y = x$, From ① $24y - 2y^2 - y^2 = 0$

$\Rightarrow 24y - 3y^2 = 0$

$\Rightarrow 3y(8 - y) = 0$

$\Rightarrow \boxed{y=0}$ or $\boxed{y=8}$

$\Rightarrow \boxed{x=0}$, $\boxed{x=8}$

If $x + y = 24$, From $\Rightarrow x = 24 - y$

$\therefore$ From (1) $24y - 2(24-y)y - y^2 = 0$

$\Rightarrow 24y - 48y + 2y^2 - y^2 = 0$

$\Rightarrow -24y + y^2 = 0$

$\Rightarrow y(y-24) = 0$

$\Rightarrow \boxed{y=0}$, $\boxed{y=24}$

$\Rightarrow \boxed{x=24}$, $\boxed{x=0}$

(i) Let $(x,y) = (0,0)$

$f_{xx} = -2y$

$f_{xy} = 24 - 2x - 2y$

$f_{yy} = -2x$

(i) Let $(x,y) = (0,0)$

$\therefore r = 0$, $s = 24$ & $t = 0$

$\therefore rt - s^2 = -576 < 0$

$\therefore$ No maxima, No minima.

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(ii) Let $(x,y) = (8,8)$

$\therefore r = -16$, $s = 24 - 16 - 16 = -8$, $t = -16$

$\therefore rt - s^2 = 256 - 64 = 192 > 0$, $r = -16 < 0$

$\therefore (x,y) = (8,8)$ is maxima and gives us maximum value.

$f(8,8) = 8 \times 8 \times [24 - [8+8]]$

$= 64 \times 8$

$= 512$

$\therefore$ Required three positive numbers are 8, 8, 8 whose sum is 24 and product is maximum.

(iii) Let $(x,y) = (0,24)$

$\therefore r = -48$, $s = 24 - 48 = -24$, $t = 0$

$rt - s^2 = 0 - 576 = -576 < 0$

$\therefore$ No maxima, No minima.

(iv) Let $(x,y) = (24,0)$

$\therefore r = 0$, $s = -24$ & $t = -48$

$\therefore rt - s^2 = 0 - 576 = -576 < 0$

$\therefore$ No maxima, No minima.

★ Lagrange's method of undetermined multiplies, to determine the extreme values of a function of n variables.

Proof: Let u be a real differentiable function of n variables $x_1, x_2, \dots, x_n$ say

$$u = \phi(x_1, x_2, x_3, \dots, x_n) \quad \text{--- (1)}$$

whose extreme values are to be determined subject to $m (< n)$ conditions (equations)

$$f_i(x_1, x_2, x_3, \dots, x_n) = 0, \quad i = 1, 2, 3, \dots, m \quad \text{--- (2)}$$

Where each f_i is differentiable.

So that only $n-m$ of n variables are independent.

For extreme values of u.

$$du = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} dx_i = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} dx_i = 0. \quad \text{--- (3)}$$

and $$df_r = \sum_{i=1}^{n} \frac{\partial f_r}{\partial x_i} dx_i = 0 \quad \text{--- } (m+3), \ r = 1, 2, 3, \dots, m$$

Multiply both sides of results (3), (4), (5), ..., (m+3) resp. by $1, \lambda_1, \lambda_2, \dots, \lambda_m$ and add all results by columns.

∴ We have the equation.

$$P_1 dx_1 + P_2 dx_2 + P_3 dx_3 + \dots + P_n dx_n = 0 \quad \text{--- (I)}$$

where $$P_i = \frac{\partial \phi}{\partial x_i} + \lambda_1 \frac{\partial f_1}{\partial x_i} + \lambda_2 \frac{\partial f_2}{\partial x_i} + \dots + \lambda_m \frac{\partial f_m}{\partial x_i}$$

$$i = 1, 2, 3, \dots, n \quad \text{--- (II)}$$

Now $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m$ used above are called undetermined multipliers and are at our choice. So we choose them so as to satisfy the following m equations

$$P_1 = 0, \ P_2 = 0, \ P_3 = 0, \dots, P_m = 0 \quad \text{--- (III)}$$

∴ The eqⁿ (I) reduces to

$$P_{m+1} dx_{m+1} + P_{m+2} dx_{m+2} + \dots + P_n dx_n = 0 \quad \text{--- (IV)}$$

Since any $n-m$ variables of n variables $x_1, x_2, \dots, x_n$ can be taken as independent values, let they be taken as $x_{m+1}, x_{m+2}, \dots, x_n$, so from the eqⁿ (IV)

$$P_{m+1} = 0, \ P_{m+2} = 0, \dots, P_n = 0 \quad \text{--- (V)}$$

Thus in virtue of choice of λ_i's, from eqⁿ (III) & (V)

$$P_1 = P_2 = P_3 = \dots = P_n = 0 \quad \text{--- (VI)}$$

These n conditions (VI) along with the given m equations to determine m unknown multipliers $\lambda_1, \lambda_2, \dots, \lambda_m$ and the values of n variables $x_1, x_2, \dots, x_n$ for which extreme value of u may exist.

★ Lagrange's Method for Extreme values of real functions of two variables

Given any function $f(x, y) = u$ and given a condition on the variables that $g(x, y) = d$ then for finding extreme values of $u = f(x, y)$ we will use this method.

Step 1 : Let $F(x, y) = f(x, y) + \lambda [g(x, y) - d]$.

λ = Lagrange's multiplier

Step 2 : Find F_x & F_y and equate it to zero.

Step 3 : Find values of x, y and (λ).

Step 4 : Find $r = u_{xx}$, $s = u_{xy}$ & $t = u_{yy}$

Find $rt - s^2$ and check whether (x, y) gives maximum values or minimum values.

(1) Find extreme values of x^2yz^6 under $x+y+z=9$.

$\to$ Let $F(x,y,z) = x^2yz^6 + \lambda[x+y+z-9]$

$\therefore F_x = 2xyz^6+\lambda$, & $F_y = x^2z^6+\lambda$ &
$F_z = 6x^2yz^5+\lambda$

Let $F_x=0$, $F_y=0$ & $F_z=0$

$\Rightarrow 2xyz^6+\lambda=0$, $x^2z^6+\lambda=0$ & $6x^2yz^5+\lambda=0$
$\Rightarrow \lambda = -2xyz^6$ —(1) $\lambda = -x^2z^6$ —(2) $\lambda = -6x^2yz^5$ —(3)

From (1) & (2) we get $\boxed{2y=x}$ $(\because x, z \neq 0)$ —(4)
From (2) & (3) we get $\boxed{z=6y}$ —(5)

Put (4) & (5) in $x+y+z=9$ we get
$2y+y+6y=9$
$\Rightarrow \boxed{y=1} \Rightarrow \boxed{x=2}$ & $\boxed{z=6}$

If we take $x, z = 0$ so $x^2yz^6 = 0$ that's not true

$\therefore$ Required point is $(2,1,6)$ which gives us the extreme value.
$f(2,1,6) = (2^2(1)(6)^6 = 186624$

(2) Divide 15 into three parts such that its product shall be maximum.

$\to$ Let x, y, z are the required numbers.
Let $f(x,y,z) = xyz$; with $x+y+z=15$.

$F(x,y,z) = f(x,y,z) + \lambda[g(x,y,z) - 15]$
$\therefore F(x,y,z) = xyz + \lambda(x+y+z-15)$

$\therefore F_x = yz+\lambda$, $F_y = xz+\lambda$ & $F_z = xy+\lambda$
Let $F_x=0$, $F_y=0$ & $F_z=0$.

$\therefore yz+\lambda=0$ —(1), $xz+\lambda=0$ —(2), $xy+\lambda=0$ —(3)
$\Rightarrow -\lambda = yz$ —(1) $-\lambda = xz$ —(2) $-\lambda = xy$ —(3)
$\Rightarrow$ From (1) & (2) we get $x=y$
From (2) & (3) we get $y=z$
$\therefore x=y=z$ put in eqn $x+y+z=15$ we get
$3x=15 \Rightarrow x=5$, $y=5$ & $z=5$

$\therefore (5,5,5)$ which gives us the extreme value
$f(5,5,5) = 125$.

(3) A rectangular box open at the top is to have the volume 108 cubic meters. What must be the its dimensions so that its total surface is minimum.

$\to$ Let l, b and h are dimensions of box.

Total surface of the box is $lb+2bh+2lh$

$\therefore f(l,b,h) = lb+2bh+2lh$ with $lbh=108$.

$F(l,b,h) = f(l,b,h) + \lambda[g(l,b,h) - 108]$

$\therefore F(l,b,h) = lb+2bh+2lh+\lambda[lbh-108]$

$\therefore$ ~~$F_x=$~~ $F_l = b+2h+\lambda bh$
$F_b = l+2h+\lambda lh$ &
$F_h = 2b+2l+\lambda lb$

Let $F_l = 0$, $F_b = 0$ & $F_h = 0$

$\therefore b + 2h + \lambda bh = 0$ — (1), $l + 2h + \lambda lh = 0$ — (2) & $2b + 2l + \lambda lb = 0$ — (3)

$\Rightarrow b + h(2 + \lambda b) = 0$

$\Rightarrow h = \dfrac{-b}{2+\lambda b}$ — (4)

$\Rightarrow l + h(2 + \lambda l) = 0$

$\Rightarrow h = \dfrac{-l}{2+\lambda l}$ — (5)

From (4) & (5) we get $\dfrac{b}{2+b\lambda} = \dfrac{l}{2+\lambda l}$

$\Rightarrow 2b + \cancel{2} bl\lambda = 2l + bl\lambda$

$\Rightarrow 2b = 2l \quad (\because b, l \neq 0)$

$\Rightarrow \boxed{b = l}$

From eqn (2) $l(1+\lambda h) + 2h = 0 \Rightarrow l = \dfrac{-2h}{1+\lambda h}$ — (6)

From eqn (3) $l(2+\lambda b) + 2b = 0 \Rightarrow l = \dfrac{-2b}{2+\lambda b}$ — (7)

$\therefore$ From eqn (6) & (7) we get $\dfrac{2h}{1+\lambda h} = \dfrac{2b}{2+\lambda b}$

$\Rightarrow 4h + 2hb\lambda = 2b + 2hb\lambda$

$\Rightarrow 4h = 2b$

$\Rightarrow \boxed{b = 2h}$

$\therefore l = 2h, \; b = 2h$

Let us put the values of l, b, h in eqn $lbh = 108$ we get $(2h)(2h)(h) = 108$

$\Rightarrow 4h^3 = 108$

$\Rightarrow h^3 = 27 \Rightarrow \boxed{h = 3\,m}$

$\Rightarrow \boxed{l = 6\,m}$ & $\boxed{b = 6\,m}$

$\therefore$ Required dimensions of box are 6m, 6m and 3m.

(4) Find the ~~minimum~~ maximum value of $x^2+y^2+z^2$ subject to the condition, $ax+by+cz-d=0$, $a^2+b^2+c^2 \neq 0, d \neq 0$

→ Here $f(x,y,z) = x^2+y^2+z^2$

$\therefore F(x,y,z) = f(x,y,z) + \lambda(ax+by+cz-d)$

$\therefore F(x,y,z) = x^2+y^2+z^2 + \lambda(ax+by+cz-d)$

$\therefore F_x = 2x + \lambda a, \quad F_y = 2y + \lambda b, \quad F_z = 2z + \lambda c$

Let $F_x = 0, \; F_y = 0, \; F_z = 0$

$\therefore 2x + \lambda a = 0 \qquad 2y + \lambda b = 0 \qquad 2z + \lambda c = 0$

$\therefore \lambda = \dfrac{-2x}{a} \qquad \lambda = \dfrac{-2y}{b} \qquad \lambda = \dfrac{-2z}{c}$

$\therefore \lambda = \dfrac{-2x}{a} = \dfrac{-2y}{b} = \dfrac{-2z}{c}$

$\Rightarrow \dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = \dfrac{x^2+y^2+z^2}{ax+by+cz} = \dfrac{x^2+y^2+z^2}{d} = \dfrac{u}{d}$

$= \dfrac{ax+by+cz}{a^2+b^2+c^2} = \dfrac{d}{a^2+b^2+c^2}$

$\therefore \dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = \dfrac{d}{a^2+b^2+c^2} = \dfrac{u}{d}$; $u = x^2+y^2+z^2$

$\Rightarrow x = \dfrac{ad}{a^2+b^2+c^2}, \quad y = \dfrac{bd}{a^2+b^2+c^2}, \quad z = \dfrac{cd}{a^2+b^2+c^2}$

$u = \dfrac{d^2}{a^2+b^2+c^2}$

$u_x = f_x =$

$\therefore (x, y, z) = \left(\dfrac{ad}{a^2+b^2+c^2}, \dfrac{bd}{a^2+b^2+c^2}, \dfrac{cd}{a^2+b^2+c^2} \right)$

gives maximum value $\dfrac{d^2}{a^2+b^2+c^2}$

(5) Prove that the extreme value of $u = a^2x^2 + b^2y^2 + c^2z^2$ subject to the conditions $x^2+y^2+z^2 = 1$, $lx+my+nz=0$ is given by $\dfrac{l^2}{a^2-u} + \dfrac{m^2}{b^2-u} + \dfrac{n^2}{c^2-u} = 0$.

$\rightarrow$ Let $F(x,y,z) = a^2x^2 + b^2y^2 + c^2z^2 + \lambda(x^2+y^2+z^2-1) + \mu(lx+my+nz)$

$F_x = 2a^2x + 2x\lambda + \mu l = 0$ —(1)
$F_y = 2b^2y + 2y\lambda + \mu m = 0$ —(2)
$F_z = 2c^2z + 2z\lambda + \mu n = 0$ —(3)

Let $x(1) + y(2) + z(3)$ we get
$2(a^2x^2+b^2y^2+c^2z^2) + 2\lambda(x^2+y^2+z^2) + \mu(lx+my+nz) = 0$

$\therefore 2u + 2\lambda = 0$

$\therefore \boxed{\lambda = -u}$ —(4)

$\therefore$ From eqⁿ (1), (2), (3) & (4)

$\therefore 2a^2x - 2ux + \mu l = 0$

$\therefore 2x(a^2-u) + \mu l = 0$

$\therefore x = -\dfrac{\mu l}{2(a^2-u)}$, $y = -\dfrac{m\mu}{2(b^2-u)}$ & $z = -\dfrac{n\mu}{2(c^2-u)}$

Substituting these values in eqⁿ $lx+my+nz=0$ we get

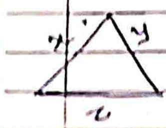
$\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$

$-\dfrac{\mu l^2}{2(a^2-u)} - \dfrac{\mu m^2}{2(b^2-u)} - \dfrac{\mu n^2}{2(c^2-u)} = 0$

$\Rightarrow -\dfrac{\mu}{2}\left[\dfrac{l^2}{a^2-u} + \dfrac{m^2}{b^2-u} + \dfrac{n^2}{c^2-u} \right] = 0$

$\Rightarrow \dfrac{l^2}{a^2-u} + \dfrac{m^2}{b^2-u} + \dfrac{n^2}{c^2-u} = 0$.

(6) Show that of all triangles, having given perimeter, the largest is an equilateral (equal sides) triangle.



$\rightarrow$ Let x, y, z be the length of sides of $\triangle ABC$. Such that $x+y+z = 2s$ = non zero constant and $u = \sqrt{s(s-x)(s-y)(s-z)} = \Delta$ Area of Triangle.

Let $F(x,y,z) = \sqrt{s(s-x)(s-y)(s-z)} + \lambda(x+y+z-2s)$

$\therefore F_x = \dfrac{-s(s-y)(s-z)}{2\sqrt{s(s-x)(s-y)(s-z)}} + \lambda = 0$.

$\Rightarrow -\dfrac{u^2}{s-x} \cdot \dfrac{1}{2u} + \lambda = 0 \quad \Rightarrow \lambda = \dfrac{u}{2(s-x)}$

$\Rightarrow 2\lambda = \dfrac{u}{s-x}$

Similarly $2\lambda = \dfrac{u}{s-y}$, $2\lambda = \dfrac{u}{s-z}$

$\therefore 2\lambda = \dfrac{u}{s-x} = \dfrac{u}{s-y} = \dfrac{u}{s-z}$

$\Rightarrow s-x = s-y = s-z$

$\Rightarrow \boxed{x = y = z = \dfrac{2s}{3}}$ $\quad$ $\therefore x+x+x = 2s \Rightarrow 3x = 2s \Rightarrow x = \dfrac{2s}{3}$

$\dfrac{x+y+z}{2} = s =$ Semi perimeter.

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(7) Show that the stationary value of $u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$ subject to $lx + my + nz = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is given by the equation $\frac{l^2a^4}{1-a^2u} + \frac{m^2b^4}{1-b^2u} + \frac{n^2c^4}{1-c^2u} = 0$.

→ Let $F(x, y, z) = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} + \lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right]$

$$+ \mu\left[lx + my + nz\right]$$

$F_x = \frac{2x}{a^4} + \frac{2x\lambda}{a^2} + \mu l = 0$ —— (1)

$F_y = \frac{2y}{b^4} + \frac{2y\lambda}{b^2} + \mu m = 0$ —— (2)

$F_z = \frac{2z}{c^4} + \frac{2z\lambda}{c^2} + \mu n = 0$ —— (3)

Let $x(1) + y(2) + z(3)$ we get

$2u + 2\lambda = 0 \Rightarrow \boxed{\lambda = -u}$ —— (4)

From (1), (2), (3) & (4) we get

$x = -\frac{\mu l a^2}{2\left(\frac{1}{a^2} - u\right)} = \frac{-\mu l a^4}{2(1-a^2u)}$

Similarly, $y = \frac{-\mu m b^4}{2(1-b^2u)}$ & $z = \frac{-\mu n c^4}{2(1-c^2u)}$

Let us put the values of x, y, z in $lx + my + nz =$

$\therefore l\left[\frac{-\mu l a^4}{2(1-a^2u)}\right] + m\left[\frac{-\mu m b^4}{2(1-b^2u)}\right] + n\left[\frac{-\mu n c^4}{2(1-c^2u)}\right] = 0$

$$\therefore \frac{u}{2}\left[\frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}\right]=0$$

$$\therefore \frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0$$

(8) Prove that the maximum or minimum distance from $(0,0,0)$ to the point on the curve of intersection $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $lx+my+nz=0$ is obtained by the eq^n $\frac{l^2a^2}{a^2-u}+\frac{m^2b^2}{b^2-u}+\frac{n^2c^2}{c^2-u}=0.$

(9) Prove that the extreme value of $u=x^2+y^2+z^2$ subject to condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $lx+my+nz=0$ is given by $\frac{l^2a^2}{a^2-u}+\frac{m^2b^2}{b^2-u}+\frac{n^2c^2}{c^2-u}=0.$

(10) In $\triangle ABC$, prove that the maximum value of $\cos A\cos B\cos C$ is $1/8$.

→ In any triangle $\triangle ABC$, $A+B+C=180$

Let $A=x$, $B=y$ and $C=z$.

$\therefore x+y+z=180^\circ$

Let $F(x,y,z)=\cos x\cos y\cos z+\lambda\cdot(x+y+z-180)$

$F_x=-\sin x\cos y\cos z+\lambda=0$ — (1)

$F_y=-\cos x\sin y\cos z+\lambda=0$ — (2)

$F_z=-\cos x\cos y\sin z+\lambda=0$ — (3)

$\therefore \lambda$ From (1) & (2) we get

$\sin(x+y)=\sin x\cos y+\cos x\sin y$

$\sin(x-y)=\sin x\cos y-\cos x\sin y$

$\cos(x+y)=\cos x\cos y-\sin x\sin y$

$\cos(x-y)=\cos x\cos y+\sin x\sin y$

$\sin x\cos y\cos z=\cos x\sin y\cos z$

$\Rightarrow \sin x\cos y\cos z-\cos x\sin y\cos z=0$

$\Rightarrow \cos z(\sin x\cos y-\cos x\sin y)=0$

$\Rightarrow \cos z=0$ or $\sin(x-y)=0$

if we take value so $\Rightarrow x-y=0$ $\qquad x-y=\pi$

we get $\cos x\cos y\cos z=0 \Rightarrow \boxed{x=y}$ ~~$t=2k\pi$~~, ~~$t\in Z$~~ $x+y+z=180$ — (4)

From eq^n (2) & (3) we get

$\cos x\sin y\cos z=\cos x\cos y\sin z$

$\Rightarrow \cos x\sin y\cos z-\cos x\cos y\sin z=0$

$\Rightarrow \cos x(\sin y\cos z-\cos y\sin z)=0$

$\Rightarrow \cos x=0$ or $\sin(y-z)=0$

$\Rightarrow y-z=0$

$\Rightarrow \boxed{y=z}$ — (5)

From (4) & (5) $x=y=z$

$\therefore x+x+x=180^\circ \Rightarrow 3x=180^\circ \Rightarrow x=60^\circ$

$\therefore \boxed{x=y=z=60^\circ}$

(11) A rectangular box open at the top is to have the surface area 108 square meters, what must be its dimensions so that its volume is maximum.

→ Suppose x,y,z are dimensions of box.

$\therefore xy+2yz+2zx=108$

$f(x,y,z)=xyz$

$\therefore F(x,y,z)=xyz+\lambda(xy+2xy+2zx-108)$.

$F_x=yz+\lambda y+2\lambda z=0$.

$F_y=xz+\lambda x+$

$\therefore F(x,y,z) = xyz + \lambda(xy + 2yz + 2zx)$

$F_x = yz + \lambda y + 2\lambda z = 0$ —— (1)

$F_y = xz + \lambda x + 2\lambda z = 0$ —— (2)

$F_z = xy + 2\lambda y + 2\lambda x = 0$ —— (3)

From eqn (1) $\lambda = -\dfrac{yz}{y+2z}$

From eqn (2) $\lambda = \dfrac{-xz}{x+2z}$

$\Rightarrow \dfrac{-yz}{y+2z} = \dfrac{-xz}{x+2z} \Rightarrow \dfrac{y}{y+2z} = \dfrac{x}{x+2z}$

$\Rightarrow y(x+2z) = x(y+2z)$

$\Rightarrow xy + 2yz = xy + 2xz$

$\Rightarrow \boxed{y = x}$ —— (4)

From (3) & (4) we get

$x^2 + 2x\lambda + 2x\lambda = 0$

$\Rightarrow x^2 + 4x\lambda = 0$

$\Rightarrow x(x + 4\lambda) = 0$

$\Rightarrow x = 0$ or $x + 4\lambda = 0$

but $x \neq 0$ $\Rightarrow x = -4\lambda \Rightarrow \boxed{x = y = -4\lambda}$

Let us put the values of y in eqn (1) we get

$-4z\lambda - 4\lambda^2 + 2z\lambda = 0$

$\Rightarrow -2z\lambda - 4\lambda^2 = 0$

$\Rightarrow -2\lambda(z + 2\lambda) = 0$

$\Rightarrow \boxed{z = -2\lambda}$

Let us put the values of x, y & z in eqn $xy + 2yz + 2xz = 108$ we get

$(-4\lambda)^2 + 2(-4\lambda)(-2\lambda) + 2(-4\lambda)(-2\lambda) = 108$

$z = 108$

$x^2 + 2($

$(2z)(2z) + 2(2z)z + 2(2z)(z)$

$4z^2 + 4z^2 + 4z^2$ $\quad 12z^2 = 108$

$z^2 = \dfrac{108}{12}$

$z^2 = 9$

$\boxed{z = 3}$

$\Rightarrow 48\lambda^2 = 108$

$\Rightarrow \lambda^2 = \dfrac{108}{48}$

$\Rightarrow \lambda = \pm 1.5$

If $\lambda = 1.5 \Rightarrow x = -6, y = -6$ & $z = -3$ (not possible)

$\therefore \boxed{\lambda = -1.5 \Rightarrow x = 6, y = 6 \text{ \& } z = 3}$

$\therefore$ Required dimensions of box are 6m, 6m & 3m.

(12) At which point of the surface $x^2 + y^2 + z^2 = 1$ whose distances from point $(2,1,3)$ is maximum.

→ Let us suppose that the required point is (x,y,z)

$\therefore f(x,y,z) = (x-2)^2 + (y-1)^2 + (z-3)^2$

$F(x,y,z) = (x-2)^2 + (y-1)^2 + (z-3)^2 + \lambda(x^2 + y^2 + z^2 - 1)$

$\therefore F_x = 2(x-2) + 2x\lambda = 0$ —— (1)

$F_y = 2(y-1) + 2y\lambda = 0$ —— (2)

$F_z = 2(z-3) + 2z\lambda = 0$ —— (3)

From (1), (2) & (3) we get

$\cancel{\lambda =}\ 1 + \lambda = \dfrac{4}{2x} = \dfrac{2}{2y} = \dfrac{6}{2z}$

$\therefore \boxed{x = 2y}$ & $\boxed{z = 3y}$

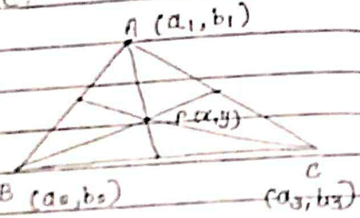
$\Rightarrow 4y^2 + y^2 + 9y^2 = 1 \Rightarrow 14y^2 = 1$

$\Rightarrow y^2 = \dfrac{1}{\sqrt{14}} \Rightarrow y = \pm\dfrac{1}{\sqrt{14}}$

$\Rightarrow x = \pm \frac{2}{\sqrt{14}}$ & $z = \pm \frac{3}{\sqrt{14}}$

$\therefore$ Required point is $\left(\pm \frac{2}{\sqrt{14}}, \pm \frac{1}{\sqrt{14}}, \pm \frac{3}{\sqrt{14}} \right)$.

(13) Prove that in any $\triangle ABC$, there exists a point P in $\triangle ABC$ such that $PA^2 + PB^2 + PC^2$ is minimum and that P is the centroid of $\triangle ABC$.



$\rightarrow$ $f(x,y) = PA^2 + PB^2 + PC^2$

$= (x-a_1)^2 + (y-b_1)^2 + (x-a_2)^2 + (y-b_2)^2 + (x-a_3)^2 + (y-b_3)^2$

$f_x = 2(x-a_1) + 2(x-a_2) + 2(x-a_3)$

$= 2(3x - (a_1+a_2+a_3))$

$f_y = 2(y-b_1) + 2(y-b_2) + 2(y-b_3)$

$= 2(3y - (b_1+b_2+b_3))$

Let $f_x = 0$ and $f_y = 0$.

$\therefore 3x - (a_1+a_2+a_3) = 0$ & $3y - (b_1+b_2+b_3) = 0$

$\Rightarrow x = \frac{a_1+a_2+a_3}{3}$ & $y = \frac{b_1+b_2+b_3}{3}$

$f_{xx} = 6$, $f_{xy} = 0$, $f_{yy} = 6$

$\therefore r = 6, s = 0$ & $t = 6$.

$\therefore rt - s^2 = 36 > 0$, $r = 6 > 0$

$\therefore (x, y) = \left(\frac{a_1+a_2+a_3}{3}, \frac{b_1+b_2+b_3}{3} \right)$ gives minimum value.

But $P = \left(\frac{a_1+a_2+a_3}{3}, \frac{b_1+b_2+b_3}{3} \right)$ is centroid.

(14) Prove that the extreme values of $u = x^2+y^2+z^2$ subject to $\Sigma ax^2 = 1$ are the roots of eq^n $(1-au)(1-bu)(1-cu) = 0$.

$\rightarrow$ $u = x^2+y^2+z^2$

$F(x,y,z) = x^2+y^2+z^2 + \lambda(ax^2+by^2+cz^2)$

$F_x = 2x + 2ax\lambda = 0$ — (1)

$F_y = 2y + 2by\lambda = 0$ — (2)

$F_z = 2z + 2cz\lambda = 0$ — (3)

From eq^n (1), (2), (3) we get

$2x(1+a\lambda) = 0$ $\qquad$ $2y(1+b\lambda) = 0$ $\qquad$ $2z(1+c\lambda) = 0$

$\therefore x = 0$ or $\lambda = -\frac{1}{a}$ $\qquad$ $\therefore y = 0$ or $\lambda = -\frac{1}{b}$ $\qquad$ $\therefore z = 0$ or $\lambda = -\frac{1}{c}$

$\therefore \lambda = -\frac{1}{a} = -\frac{1}{b} = -\frac{1}{c} = -\left(\frac{x^2+y^2+z^2}{ax^2+by^2+cz^2} \right) = -u$

$\therefore \boxed{\lambda = -u}$ — (4)

From (1), (2), (3) & (4) $u = \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$

Let us put the value of u in $(1-au)(1-bu)(1-cu)$

$\therefore (1-au)(1-bu)(1-cu) = (1-a\times\frac{1}{a})(1-b\times\frac{1}{b})(1-c\times\frac{1}{c})$

$= 0$

$\therefore$ Extreme value of $u = x^2+y^2+z^2$ are the roots of eq^n $(1-au)(1-bu)(1-cu) = 0$.

(15) Prove that $f(x,y) = \sin^2 x + \sin^2 y$, where $x+y = a$ has minimum value $1-\cos a$ and maximum value $1+\cos a$.

$\rightarrow$ $f_x = \sin 2x$ & $f_y = \sin 2y$

Let $f_x = 0$ & $f_y = 0$

$\therefore \sin 2x = \sin 2y$

$\Rightarrow 2x = 2y + 2k\pi,\ k \in \mathbb{Z}$

$\Rightarrow x = y + k\pi,\ k \in \mathbb{Z}$

but $x+y = a \Rightarrow 2y + k\pi = a \Rightarrow y = \frac{a-k\pi}{2} = \frac{a}{2} - \frac{k\pi}{2}$

$\Rightarrow x = \frac{a}{2} + \frac{k\pi}{2}$

$f_{xx} = 2\cos 2x$, $f_{xy} = 0$, $f_{yy} = 2\cos 2y$

$\therefore r = 2\cos 2\left(\frac{a}{2} + \frac{k\pi}{2}\right) = 2\cos(a+k\pi) = \pm 2\cos a$

$s = 0$

$t = 2\cos\left(\frac{a}{2} - \frac{k\pi}{2}\right) = 2\cos(a-k\pi) = \pm 2\cos a$

$r = 2\cos a,\ s = 0,\ t = 2\cos a$ If k is even

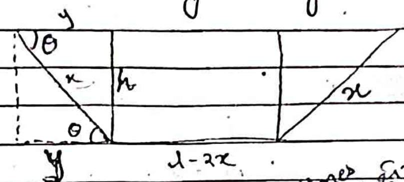
$r = -2\cos a,\ s = 0,\ t = -2\cos a$ If k is odd

$rt - s^2 = 4\cos^2 a > 0$, $r = 2\cos a > 0 \Rightarrow$ minimum

bases - x

trough - θ

(16) A rectangular strip $l \times b$ of metal is bent up at the sides to form a trough without ends. Find the width of the side bases and the angle through the side must bent so that the trough may have a maximum capacity.



$\rightarrow$ $\sin\theta = \frac{h}{x} \Rightarrow h = x\sin\theta$

$\cos\theta = \frac{y}{x} \Rightarrow y = x\cos\theta$

2 times Δ s

$\rightarrow$ Volume of trough $= (l-2x)\,bh + 2\times\frac{1}{2}\,yhb$

$= (l-2x)\,b\,x\sin\theta + x^2\cos\theta\sin\theta\, b$

$= b(lx-2x^2)\sin\theta + \frac{x^2}{2}b\sin 2\theta$

$f(x,\theta) = b(lx-2x^2)\sin\theta + \frac{x^2}{2}b\sin 2\theta$

$f_x = b(l-4x)\sin\theta + xb\sin 2\theta = 0$ —(1)

$f_\theta = b(lx-2x^2)\cos\theta + x^2 b\cos 2\theta = 0$ —(2)

From (1) we get

$\cancel{b}(l-4x)\cancel{\sin\theta} = -2\cancel{b}x\cancel{\sin\theta}\cos\theta$

$\therefore \cos\theta = \frac{4x-l}{2x}$

$\cos 2\theta = \ldots$

From (a) we get

$b(lx-2x^2)\cos\theta + bx^2(2\cos^2\theta - 1) = 0$

$\therefore b(lx-2x^2)\dfrac{4x-l}{2x} + 2bx^2\dfrac{(4x-l)^2}{4x^2} - bx^2 = 0$

$\therefore \dfrac{bx(l-2x)(4x-l)}{2x} + \dfrac{b(4x-l)^2}{2} - bx^2 = 0$

$\therefore b\left[\dfrac{(l-2x)(4x-l)}{2} + \dfrac{(4x-l)^2}{2}\right] = bx^2$

$\therefore \dfrac{4x-l}{2}\left[(l-2x)+(4x-l)\right] = x^2$

$\therefore \dfrac{4x-l}{2}(2x) = x^2$

$\therefore 4x^2 - lx - x^2 = 0$

$\therefore 3x^2 - lx = 0$

$\therefore x(3x-l) = 0$

$\therefore x = 0$ or $x = \dfrac{l}{3}$

but $x \neq 0$ $\therefore x = l/3$

When $x = \dfrac{l}{3} \Rightarrow \cos\theta = \dfrac{\frac{4}{3}l - l}{2\frac{l}{3}} = \dfrac{l/3}{2l/3} = \dfrac{1}{2}$

$\Rightarrow \boxed{\theta = 60^\circ}$

$r = f_{xx} = -4b\sin\theta + b\sin 2\theta$

$t = f_{\theta\theta} = -b(lx-2x^2)\sin\theta - 2bx^2\sin 2\theta$

$s = f_{x\theta} = b(l-4x)\cos\theta + 2bx\cos 2\theta$

$\sin 120 = \sin(90+30) = \cos 30 = \sqrt{3}/2$

$\cos 120 = \cos(90+30) = -\sin 30 = -1/2$

$\therefore r = -4b\dfrac{\sqrt{3}}{2} + b\dfrac{\sqrt{3}}{2} = -3\sqrt{3}\dfrac{b}{2}$

$s = b\left(l - \dfrac{4l}{3}\right)\dfrac{1}{2} - 2b\dfrac{l}{3}\dfrac{1}{2}$

$= -\dfrac{bl}{6} - \dfrac{2bl}{6}$

$s = -\dfrac{bl}{2}$

$t = -b\left(\dfrac{2l^2}{9} - \dfrac{2l^2}{9}\right)\dfrac{\sqrt{3}}{2} - 2b\dfrac{l^2}{9}\dfrac{\sqrt{3}}{2}$

$= -\dfrac{bl^2}{9}\dfrac{\sqrt{3}}{2} - \dfrac{bl^2}{9}\sqrt{3}$

$= -\dfrac{\sqrt{3}bl^2}{6}$

$\therefore rt - s^2 = \dfrac{(-3\sqrt{3})b}{2}\left(\dfrac{-\sqrt{3}bl^2}{6}\right) - \dfrac{b^2l^2}{4}$

$= \dfrac{9b^2l^2}{12} - \dfrac{b^2l^2}{4}$

$= \dfrac{3b^2l^2 - b^2l^2}{4}$

$= \dfrac{b^2l^2}{2} > 0$ & $r = -\dfrac{3\sqrt{3}}{2} < 0$

$\therefore$ Maximum capacity obtain when $\boxed{x = \dfrac{l}{3}}$ &

$\boxed{\theta = \dfrac{\pi}{3} = 60^\circ}$

(15) Prove that $f(x,y) = \sin^2 x + \sin^2 y$, where $x+y=a$ has minimum value $1-\cos a$ and maximum value $1+\cos a$.

$\rightarrow f_x = \sin 2x = 0$ & $f_y = \sin 2y = 0$

$\Rightarrow \sin 2x = \sin 2y$

$\Rightarrow 2x = 2y + 2k\pi, \; k \in \mathbb{Z}$

$\Rightarrow x = y + k\pi, \; k \in \mathbb{Z}$

$x+y = a \Rightarrow 2y + k\pi = a \Rightarrow \boxed{y = \frac{a}{2} - \frac{k\pi}{2}}$

& $\boxed{x = \frac{a}{2} + \frac{k\pi}{2}}$

$f_{xx} = 2\cos 2x \Rightarrow r = 2\cos 2\left(\frac{a}{2} + \frac{k\pi}{2}\right) = 2\cos(a + k\pi)$

$= \pm 2\cos a$

S | A
T | C

$f_{xy} = 0 \Rightarrow s = 0$

$f_{yy} = 2\cos 2y \Rightarrow t = 2\cos 2\left(\frac{a}{2} - \frac{k\pi}{2}\right) = 2\cos(a - k\pi)$

$= \pm 2\cos a$

If k is even ($k = 2n, n \in \mathbb{Z}$)

$r = 2\cos a, \; s = 0, \; t = 2\cos a$

$\therefore rt - s^2 = 4\cos^2 a - 0 = 4\cos^2 a > 0$

$\therefore \left(\frac{a}{2} + \frac{k\pi}{2}, \frac{a}{2} - \frac{k\pi}{2}\right)$ gives minimum value.

$f\left(\frac{a}{2} + \frac{k\pi}{2}, \frac{a}{2} - \frac{k\pi}{2}\right) = f\left(\frac{a}{2} + \frac{2n\pi}{2}, \frac{a}{2} - \frac{2n\pi}{2}\right)$

$= f\left(\frac{a}{2} + n\pi, \frac{a}{2} - n\pi\right)$

$= \sin^2\left(\frac{a}{2} + n\pi\right) + \sin^2\left(\frac{a}{2} - n\pi\right)$

$= \sin^2 \frac{a}{2} + \sin^2 \frac{a}{2}$

$= \boxed{2\sin^2 \frac{a}{2}}$

$= \boxed{1 - \cos a}$

$\therefore$ Minimum value is $1 - \cos a$.

If k is odd ($k = 2n+1, n \in \mathbb{Z}$)

$r = -2\cos a, \; s = 0$ & $t = -2\cos a$

$\therefore rt - s^2 = 4\cos^2 a - 0 = 4\cos^2 a > 0, \; r = -2\cos a < 0$

$\therefore \left(\frac{a}{2} + \frac{(2n+1)\pi}{2}, \frac{a}{2} - \frac{(2n+1)\pi}{2}\right)$ gives maximum value.

$\therefore f\left(\frac{a}{2} + \frac{(2n+1)\pi}{2}, \frac{a}{2} - \frac{(2n+1)\pi}{2}\right) = \sin^2\left(\frac{a}{2} + \frac{(2n+1)\pi}{2}\right) + \sin^2\left(\frac{a}{2} - \frac{(2n+1)\pi}{2}\right)$

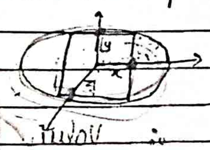
$= \cos^2 \frac{a}{2} + \cos^2 \frac{a}{2}$

$= \boxed{2\cos^2 \frac{a}{2}}$

$= \boxed{1 + \cos a}$

$\therefore$ Maximum value is $1 + \cos a$.

(17) Prove that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$.



$\rightarrow \boxed{f(x,y,z) = 2x \cdot 2y \cdot 2z}$

$= 8xyz$

$$\therefore F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$F_x = 8yz + \frac{2x\lambda}{a^2} = 0 \quad \text{---(1)}$$

$$F_y = 8xz + \frac{2y\lambda}{b^2} = 0 \quad \text{---(2)}$$

$$F_z = 8xy + \frac{2z\lambda}{c^2} = 0 \quad \text{---(3)}$$

From (1), (2) and (3) we get

$$\lambda = \frac{-4yza^2}{x}, \quad \lambda = \frac{-4xzb^2}{y} \quad \& \quad \lambda = -\frac{4xyc^2}{z}$$

$$\therefore \lambda = \frac{-4xyza^2}{x^2} = \frac{-4xyzb^2}{y^2} = \frac{-4xyzc^2}{z^2}$$

$$\therefore \frac{1}{\lambda} = \frac{x^2}{-4xyza^2} = \frac{y^2}{-4xyzb^2} = \frac{z^2}{-4xyzc^2}$$

$$\therefore -\frac{1}{4xyz}\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = \frac{3}{\lambda}$$

$$\therefore -\frac{1}{4xyz} = \frac{3}{\lambda}$$

$$\therefore \lambda = -12xyz \quad \text{---(4)}$$

From (1) & (4) we get $8yz - \frac{24x^2yz}{a^2} = 0$

$$\therefore a^2 = 3x^2 \Rightarrow x = \frac{a}{\sqrt{3}}$$

Similarly $y = \frac{b}{\sqrt{3}}$ & $z = \frac{c}{\sqrt{3}}$

$$\therefore \text{Volume} = 8xyz = \frac{8abc}{\sqrt{3}\sqrt{3}\sqrt{3}} = \frac{8abc}{3\sqrt{3}}$$

If we take volume xyz then eq^n of ellipsoid may becomes $\left(\frac{x}{2}\right)^2\frac{1}{a^2} + \left(\frac{y}{2}\right)^2\frac{1}{b^2} + \left(\frac{z}{2}\right)^2\frac{1}{c^2} = 1$

$$\therefore \frac{x^2}{4a^2} + \frac{y^2}{4b^2} + \frac{z^2}{4c^2} = 1$$

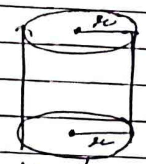
Because $x/2, y/2, z/2$ are end points of ~~cube~~ parallelopiped and lies on ellipsoid hence the eq^n.

(18) Find the dimensions of a cylindrical box which is open at top and whose surface area is 60 and its volume becomes maximum.

Sol^n ~~For~~, l - length, h - height, r = radius

Total surface area

$$= \pi r^2 + \pi r^2 + 2\pi rh$$



Total surface are of a cylindrical box which is open at top is

$$= \pi r^2 + 2\pi rh$$

Total volume of cylindrical box

$$= \pi r^2 h$$

Vector calculus

* Scalar function :

A function whose images are scalars only then that function is called scalar function.

e.g. $f(x,y) = x^2 + y^2 + 2xy$.

A scalar which is uniquely determined on some set i.e. at every point of a region, is said to be scalar point function. Temperature at every point of atmosphere is defined to be scalar function.

* Vector function :

A function whose images are vector is called vector function.

e.g. $f(x,y,z) = x^2\hat{i} + y^2\hat{j} + z^2\hat{k} = (x^2, y^2, z^2)$

A vector which is uniquely determined on some set i.e. at every point of a region, by $f(\vec{r})$.

* Vector operator (∇) (nabla) :

In cartesian co-ordinate system

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \nabla$$

is called a vector operator.

* Gradient of scalar function :

If $\phi(x,y,z)$ is a scalar function defined on an open region S of $\mathbb{R}^3$ whose partial derivatives exists, then vector function $\left(\frac{\partial\phi}{\partial x}(x,y,z), \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$ is called gradient of $\phi(x,y,z)$ and it is denoted by $\nabla\phi$.

∇f = Gradient of any scalar point function (i.e. a vector) to represent the normal (Gr) vector to the surface

(1) Find grad $f = \nabla f$ for $f(x,y,z) = 3xy^2 + 2x^2yz$ at a point $(2,1,-1)$.

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$= (3y^2 + 4xyz,\ 6xy + 2x^2z,\ 2x^2y)$

$\nabla f(2,1,-1) = (3-8,\ 12-8,\ 8)$

$= (-5, 4, 8)$

★ Properties : Thm 1

(1) grad $(f+g)$ = grad f + grad g

Proof: L.H.S. $= \left(\frac{\partial(f+g)}{\partial x}, \frac{\partial(f+g)}{\partial y}, \frac{\partial(f+g)}{\partial z}\right)$

$= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}, \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right)$

$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) + \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$

= grad f + grad g

= R.H.S.

(2) grad $(fg) = g \cdot$ grad $f + f \cdot$ grad g

Proof: L.H.S. = grad (fg)

$= \left(\frac{\partial}{\partial x}(fg), \frac{\partial}{\partial y}(fg), \frac{\partial}{\partial z}(fg)\right)$

$= \left(g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}, g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}, g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z}\right)$

$= g\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) + f\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$

$= g \cdot$ grad $f + f \cdot$ grad g

= R.H.S.

(3) grad $\left(\frac{f}{g}\right) = \frac{g \cdot \text{grad } f - f \cdot \text{grad } g}{g^2}$

Proof: L.H.S. = grad $\left(\frac{f}{g}\right)$

$= \left(\frac{\partial}{\partial x}(f/g), \frac{\partial}{\partial y}(f/g), \frac{\partial}{\partial z}(f/g)\right)$

$= \left(\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}, \frac{g\frac{\partial f}{\partial y} - f\frac{\partial g}{\partial y}}{g^2}, \frac{g\frac{\partial f}{\partial z} - f\frac{\partial g}{\partial z}}{g^2}\right)$

$= \frac{1}{g^2}\left[g\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) - f\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)\right]$

$= \frac{1}{g^2}\left[g \cdot \text{grad } f - f \cdot \text{grad } g\right]$

$= \frac{g \cdot \text{grad } f - f \cdot \text{grad } g}{g^2}$

= R.H.S.

Examples:

(1) (HW) If $\bar{r} = (x,y,z)$; $|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$, then prove that $\nabla r^m = m r^{m-2}\,\bar{r}$.

Proof: $\bar{r} = (x,y,z)$

$\therefore |\bar{r}| = \sqrt{x^2+y^2+z^2} = r$.

$r = f(x,y,z)$

$$\nabla r^m = \left(\frac{\partial(r^m)}{\partial x}, \frac{\partial(r^m)}{\partial y}, \frac{\partial(r^m)}{\partial z}\right)$$

$$= \left(\frac{\partial(r^m)}{\partial r}\frac{\partial r}{\partial x}, \frac{\partial(r^m)}{\partial r}\frac{\partial r}{\partial y}, \frac{\partial(r^m)}{\partial r}\frac{\partial r}{\partial z}\right)$$

$$= \left(mr^{m-1}\frac{x}{r}, mr^{m-1}\frac{y}{r}, mr^{m-1}\frac{z}{r}\right) \qquad \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r} = x/r$$

$$= mr^{m-2}(x,y,z)$$

$$\boxed{\nabla r^m = mr^{m-2}\bar{r}}$$

Hence proved.

(2) If $\bar{r} = (x,y,z)$ then prove that $grad(\frac{1}{r}) = -\frac{\bar{r}}{r^3}$ where $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$. (Also prove that $\nabla^2(\frac{1}{r}) = 0$)

→ $grad\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right)$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \left(\frac{\partial(\frac{1}{r})}{\partial x}, \frac{\partial(\frac{1}{r})}{\partial y}, \frac{\partial(\frac{1}{r})}{\partial z}\right)$$

let $\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x}$

$$= \frac{-1}{r^2}\frac{x}{r}$$

$$= \frac{-x}{r^3}$$

Similarly $\frac{\partial(\frac{1}{r})}{\partial y} = \frac{-y}{r^3}$ and $\frac{\partial(\frac{1}{r})}{\partial z} = \frac{-z}{r^3}$

$\therefore grad\left(\frac{1}{r}\right) = \left(-\frac{x}{r^3}, -\frac{y}{r^3}, -\frac{z}{r^3}\right)$

$\nabla^2\phi = \nabla\cdot(\nabla\phi)$ or $grad(grad(1/r)) = 0$ (2013)

dot product,

so, we get scalars ②

$$= \frac{-1}{r^3}(x,y,z) \qquad \nabla\left(grad\left(\frac{1}{r}\right)\right) = \nabla\left(-\frac{r}{r^3}\right)$$

$$= \frac{-1}{r^3}\bar{r} \qquad \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(\frac{-x}{r^3}, \frac{-y}{r^3}, \frac{-z}{r^3}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{-x}{r^3}\right) = -\left(\frac{x3r^2\frac{x}{r} - r^3}{r^6}\right) = -\left(\frac{3x^2r - r^3}{r^6}\right)$$

addition of

$$-\left(\frac{3x^2r-r^3}{r^6}\right) + -\left(\frac{3y^2r-r^3}{r^6}\right) - \left(\frac{3z^2r-r^3}{r^6}\right) = 0$$

(3) If $\bar{r} = (x,y,z)$ then prove that $\nabla^2(f(r)) = f''(r) + \frac{2}{r}f'(r)$.

→ $\nabla^2(f(r)) = \nabla\cdot(\nabla(f(r)))$

$$\nabla(f(r)) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(f(r))$$

$$= \left(\frac{\partial f(r)}{\partial x}, \frac{\partial f(r)}{\partial y}, \frac{\partial f(r)}{\partial z}\right)$$

$$\frac{\partial f(r)}{\partial x} = \frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}$$

Similarly $\frac{\partial f(r)}{\partial y} = f'(r)\frac{y}{r}$ and $\frac{\partial f(r)}{\partial z} = f'(r)\frac{z}{r}$

$$\nabla(f(r)) = \left(f'(r)\frac{x}{r}, f'(r)\frac{y}{r}, f'(r)\frac{z}{r}\right) \quad \text{—(1)}$$

$$\nabla^2(f(r)) = \nabla\cdot(\nabla(f(r)))$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(f'(r)\frac{x}{r}, f'(r)\frac{y}{r}, f'(r)\frac{z}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(f'(r)\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(f'(r)\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(f'(r)\frac{z}{r}\right)$$

let $\frac{\partial}{\partial x}\left(f'(r)\frac{x}{r}\right) = \frac{\partial}{\partial x}(f'(r))\frac{x}{r} + f'(r)\frac{\partial}{\partial x}\frac{x}{r}$ —(2)

$$= f''(r)\frac{x^2}{r^2} + f'(r)\frac{r - x(\frac{x}{r})}{r^2}$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\frac{(r^2-x^2)}{r^3} \quad \text{—③}$$

Similarly

$$\frac{\partial}{\partial y}\left(f'(r)\frac{y}{r}\right) = \frac{y^2}{r^2} f''(r) + f'(r)\,\frac{(r^2-y^2)}{r^3} \quad \text{—(4) and}$$

$$\frac{\partial}{\partial z}\left(f'(r)\frac{z}{r}\right) = \frac{z^2}{r^2} f''(r) + f'(r)\,\frac{(r^2-z^2)}{r^3} \quad \text{—(5)}$$

Using (3), (4), (5) in (2) we get

$$\nabla^2(f(r)) = \frac{1}{r^2} f''(r)\,(x^2+y^2+z^2) + \frac{f'(r)}{r^3}\,(3r^2-r^2)$$

$$\boxed{\nabla^2(f(r)) = f''(r) + \frac{2}{r} f'(r)}$$

(4) If p and q are constant vectors and $\bar{r} = (x,y,z)$, then prove that grad $[\bar{r}, p, q] = p \times q$.

→ Let $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$

$$[r\ p\ q] = \begin{vmatrix} x & y & z \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

$$= x(p_2q_3 - p_3q_2) - y(p_1q_3 - p_3q_1) + z(p_1q_2 - p_2q_1)$$

$$\text{grad}[r\ p\ q] = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{bmatrix} x(p_2q_3 - p_3q_2) \\ -y(p_1q_3 - p_3q_1) + \\ z(p_1q_2 - p_2q_1) \end{bmatrix}$$

$$= (p_2q_3 - p_3q_2,\ -(p_1q_3 - p_3q_1),\ p_1q_2 - p_2q_1)$$

$$= p \times q$$

(5)* If $f(x,y,z) = 3x^2y^2z + 4xy^2z^2$, then prove that grad $f_{(1,1,1)} = (10, 14, 11)$.

→ grad $f = \nabla f$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(3x^2y^2z + 4xy^2z^2)$$

$$\cancel{= (6x + 4y}$$

$$= (6xy^2z + 4y^2z^2,\ 6x^2yz + 8xyz^2,\ 3x^2y^2 + 8xy^2z)$$

$\therefore$ grad $f_{(1,1,1)} = (10, 14, 11)$.

* **Divergence of a vector function in $\mathbb{R}^3$:**

If f is a differentiable vector function defined on an open sphere (set) of $\mathbb{R}^3$, whose partial derivatives exists, then the scalar $\nabla \cdot f$ is called divergence of f and it is denoted by div f.

If $f = (f_1, f_2, f_3)$ then,

$$\boxed{\text{div } f = \nabla \cdot f}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (f_1, f_2, f_3)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Theorem: If f and g are two differentiable vector functions, then div $(f+g)$ = div f + div g.

Proof: div $(f+g) = \nabla \cdot (f+g)$ $\quad$ $f = (f_1, f_2, f_3)$, $g = (g_1, g_2, g_3)$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (f_1+g_1, f_2+g_2, f_3+g_3)$$

$$= \frac{\partial}{\partial x}(f_1+g_1) + \frac{\partial}{\partial y}(f_2+g_2) + \frac{\partial}{\partial z}(f_3+g_3)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial g_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial g_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial g_3}{\partial z}$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}$$

$div(f+g) = div\ f + div\ g$

(1) If ϕ is a scalar point functions and f and g are is differentiable vector point function then prove that $div(\phi f) = \phi\, div\ f + f \cdot grad\ \phi$

→ Let $f = (f_1, f_2, f_3) \Rightarrow \phi f = (\phi f_1, \phi f_2, \phi f_3)$

$\therefore div(\phi f) = \nabla \cdot (\phi f)$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (\phi f_1, \phi f_2, \phi f_3)$$

$$= \left(\frac{\partial(\phi f_1)}{\partial x} + \frac{\partial(\phi f_2)}{\partial y} + \frac{\partial(\phi f_3)}{\partial z}\right)$$

$$= \frac{\partial \phi}{\partial x} f_1 + \phi \frac{\partial f_1}{\partial x} + \frac{\partial \phi}{\partial y} f_2 + \phi \frac{\partial f_2}{\partial y} + \frac{\partial \phi}{\partial z} f_3 + \frac{\partial f_3}{\partial z}\phi$$

$$= \phi\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) + \left(f_1 \frac{\partial \phi}{\partial x} + f_2 \frac{\partial \phi}{\partial y} + f_3 \frac{\partial \phi}{\partial z}\right)$$

$$= \phi\, div\ f + (f_1, f_2, f_3) \cdot \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$

$$\boxed{div(\phi f) = \phi\, div\ f + f \cdot grad\ \phi}$$

* Curl of vector point function in $\mathbb{R}^3$.

If f is a differentiable vector point function defined on an open sphere (set) s of $\mathbb{R}^3$, whose partial derivatives exists then the vector point function $\nabla \times f$ is called the curl of f and denoted by curl f

$$\boxed{curl\ f = \nabla \times f}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (f_1, f_2, f_3)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

(1) Div (curl f) =

→ Div (curl f) $= \nabla \cdot (\nabla \times f)$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} + \frac{\partial^2 f_1}{\partial y \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y}$$

$$= 0$$

(2) curl (grad f) =

→ curl (grad f) $= \nabla \times (grad\ f)$

$$= \nabla \times \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= (0,0,0)$$

$$= \bar{0}$$

(5) If $\bar{r} = (x,y,z)$ & $r = |\bar{r}|$ then find that ~~div(r^m r̄)~~ $\operatorname{div}(r^m \bar{r})$

→ $\operatorname{div}(r^m \bar{r}) = \nabla \cdot (r^m \bar{r})$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (r^m x, r^m y, r^m z)$$

$$= \frac{\partial}{\partial x}(r^m x) + \frac{\partial}{\partial y}(r^m y) + \frac{\partial}{\partial z}(r^m z)$$

$$= r^m + mr^{m-1}\frac{x^2}{r} + r^m + mr^{m-1}\frac{y^2}{r} + r^m + mr^{m-1}\frac{z^2}{r}$$

$$= 3r^m + mr^{m-2}(x^2+y^2+z^2)$$

$$= 3r^m + mr^m$$

$$\boxed{\operatorname{div}(r^m \bar{r}) = r^m (3+m)}$$

Gradient of a scalar point function

In math's, the gradient is a generalization of the usual concept of derivative of a function in several dimensions. If $f(x_1, \dots, x_n)$ is a differentiable scalar valued funᶜ of standard cartesian coordinates in Euclidean space, its gradient is the vector whose components are the n partial derivatives of f. It is thus a vector valued function.

Similarly to the usual derivative, the gradient represents the slope of the tangent of the graph of th function. More precisely, the gradient points in the direction of the greatest rate of increase of the funᶜ and its magnitude is the slope of the graph in that direction. The components of the gradient in coordinat are the coefficients of the variables in the eqⁿ of the tangent space to the graph. This characterizing proper of the gradient allow it to be defined independently of a choice of coordinate system, as a vector field whose components in a coordinate system will transform wh going from one coordinate system to another.

$r^{m-1}\frac{z^2}{r}$ 7 2013

(6) $\operatorname{div}(f \times g) = g \cdot \operatorname{curl} f - f \cdot \operatorname{curl} g$

→ ~~div~~ Let $f = (f_1, f_2, f_3)$ & $g = (g_1, g_2, g_3)$

$\operatorname{div}(f \times g) = \nabla \cdot (f \times g)$

$$f \times g = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$= (f_1 g_2 - g_1 f_2, f_3 g_1 - f_1 g_3, f_1 g_2 - g_1 f_2)$$

$\therefore \operatorname{div}(f\times g) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(f_2g_3-g_2f_3,\ f_3g_1-f_1g_3,\ f_1g_2-g_1f_2\right)$

$= \frac{\partial}{\partial x}(f_2g_3-g_2f_3) + \frac{\partial}{\partial y}(f_3g_1-f_1g_3) + \frac{\partial}{\partial z}(f_1g_2-g_1f_2)$

$= \frac{\partial}{\partial x}(f_2g_3) - \frac{\partial}{\partial x}(g_2f_3) + \frac{\partial}{\partial y}(f_3g_1) - \frac{\partial}{\partial y}(f_1g_3)$

$+ \frac{\partial}{\partial z}(f_1g_2-g_1f_2)$

$= f_2\frac{\partial g_3}{\partial x} + g_3\frac{\partial f_2}{\partial x} - f_3\frac{\partial g_2}{\partial x} - g_2\frac{\partial f_3}{\partial x} + g_1\frac{\partial f_3}{\partial y} + f_3\frac{\partial g_1}{\partial y}$

$- g_3\frac{\partial f_1}{\partial y} - f_1\frac{\partial g_3}{\partial y} - g_2\frac{\partial f_1}{\partial z} + f_2\frac{\partial g_2}{\partial z} - g_1\frac{\partial f_2}{\partial z} - f_2\frac{\partial g_1}{\partial z}$

$= g_1\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_3}{\partial z}\right) + g_2\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + g_3\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$

$+ f_1\left(-\frac{\partial g_2}{\partial z} + \frac{\partial g_3}{\partial y}\right) - f_2\left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right) - f_3\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right)$

$= (g_1, g_2, g_3)\cdot\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$

$- (f_1, f_2, f_3)\cdot\left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial y}, \frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}, \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right)$

$= g\cdot \text{curl } f - f\cdot \text{curl } g$

(7) $\text{curl}(\phi f) = \phi \text{ curl } f + (\text{grad } \phi)\times f$

→ $\text{curl}(\phi f) = \nabla\times\phi f$

suppose $f = (f_1, f_2, f_3)$.

$\therefore$ curl ϕf

$= \nabla\times(\phi f_1, \phi f_2, \phi f_3)$

$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\times(\phi f_1, \phi f_2, \phi f_3)$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi f_1 & \phi f_2 & \phi f_3 \end{vmatrix}$

$= \left(\frac{\partial}{\partial y}\phi f_3 - \frac{\partial}{\partial z}\phi f_2, \frac{\partial}{\partial z}\phi f_1 - \frac{\partial}{\partial x}\phi f_3, \frac{\partial}{\partial x}\phi f_2 - \frac{\partial}{\partial y}\phi f_1\right)$

$= \Big(\phi\frac{\partial f_3}{\partial y} + f_3\frac{\partial\phi}{\partial y} - \phi\frac{\partial f_2}{\partial z} - f_2\frac{\partial\phi}{\partial z},$

$\phi\frac{\partial f_1}{\partial z} + f_1\frac{\partial\phi}{\partial z} - \phi\frac{\partial f_3}{\partial x} - f_3\frac{\partial\phi}{\partial x},$

$\phi\frac{\partial f_2}{\partial x} + f_2\frac{\partial\phi}{\partial x} - \phi\frac{\partial f_1}{\partial y} - f_1\frac{\partial\phi}{\partial y}\Big)$

$= \left(\phi\frac{\partial f_3}{\partial y} - \phi\frac{\partial f_2}{\partial z}, \phi\frac{\partial f_1}{\partial z} - \phi\frac{\partial f_3}{\partial x}, \phi\frac{\partial f_2}{\partial x} - \phi\frac{\partial f_1}{\partial y}\right)$

$+ \left(f_3\frac{\partial\phi}{\partial y} - f_2\frac{\partial\phi}{\partial z}, f_1\frac{\partial\phi}{\partial z} - f_3\frac{\partial\phi}{\partial x}, f_2\frac{\partial\phi}{\partial x} - f_1\frac{\partial\phi}{\partial y}\right)$

$= \phi\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$

$+ \left(f_3\frac{\partial\phi}{\partial y} - f_2\frac{\partial\phi}{\partial z}, f_1\frac{\partial\phi}{\partial z} - f_3\frac{\partial\phi}{\partial x}, f_2\frac{\partial\phi}{\partial x} - f_1\frac{\partial\phi}{\partial y}\right)$

$= \phi \text{ curl } f + (\text{grad } \phi)\times f$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

(8) $\operatorname{div}(\operatorname{grad}\phi) = \nabla^2\phi = \nabla.(\nabla\phi)$

$\Rightarrow$ $\operatorname{div}(\operatorname{grad}\phi) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$

$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$

$= \nabla^2\phi$

(9) If $\bar{r} = (x, y, z)$, $|\bar{r}| = r$ and $\bar{a} = (a_1, a_2, a_3)$ (constant vector), then

(i) $\nabla(\bar{a}\cdot\bar{r}) = \bar{a}$

(ii) $\nabla\cdot(\bar{a}\times\bar{r}) = \operatorname{div}(\bar{a}\times\bar{r}) = \bar{0}$ ✗ | 0 ✓

(iii) $\nabla\times(\bar{a}\times\bar{r}) = \operatorname{curl}(\bar{a}\times\bar{r}) = 2\bar{a}$

(iv) $(\bar{a}\cdot\nabla)\bar{r} = \bar{a}$

(i) $\nabla(\bar{a}\cdot\bar{r}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$

$= (a_1, a_2, a_3)$

$= \bar{a}$

(ii) $\nabla\cdot(\bar{a}\times\bar{r}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(a_2z - a_3y, a_3x - a_1z, a_1y - a_2x)$

$= (0+0+0)$

$= \cancel{\bar{0}}\ 0$

(iii) $\nabla\times(\bar{a}\times\bar{r}) = \nabla\times((a_1, a_2, a_3)\times(x, y, z))$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2z - a_3y & a_3x - a_1z & a_1y - a_2x \end{vmatrix}$$

$= \Big(\frac{\partial}{\partial y}(a_1y - a_2x) - \frac{\partial}{\partial z}(a_3x - a_1z),$

$\frac{\partial}{\partial z}(a_2z - a_3y) - \frac{\partial}{\partial x}(a_1y - a_2x),$

$\frac{\partial}{\partial x}(a_3x - a_1z) - \frac{\partial}{\partial y}(a_2z - a_3y)\Big)$

$= (a_1 + a_1, a_2 + a_2, a_3 + a_3)$

$= 2(a_1, a_2, a_3)$

$= 2\bar{a}$.

(iv) $(\bar{a}\cdot\nabla)\bar{r} = \left((a_1, a_2, a_3)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\right)(x, y, z)$.

$= \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)(x, y, z)$

$= (a_1 + 0 + 0,\ 0 + a_2 + 0,\ 0 + 0 + a_3)$

$= (a_1, a_2, a_3)$

$= \bar{a}$

$(\nabla.\bar{a})\bar{r} = \left[\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(a_1, a_2, a_3)\right](x, y, z)$

$= \left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}\right)(x, y, z)$

(10) If $f = \text{grad}\,(x^3+y^3+z^3-3xyz)$, then prove that $\text{grad}\,(\text{div}\, f) = (6,6,6)$ and $\text{grad}\,(\text{curl}\, f) = 0 \rightarrow$? Invalid

→ $f = \text{grad}\,(x^3+y^3+z^3-3xyz)$

$= \nabla\,(x^3+y^3+z^3-3xyz)$

$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$

$= (3x^2-3yz,\ 3y^2-3xz,\ 3z^2-3xy)$

$\text{div}\, f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(3x^2-3yz, 3y^2-3xz, 3z^2-3xy)$

$= (6x+6y+6z)$

$\text{grad}\,(\text{div}\, f) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)(6x+6y+6z)$

$= (6+0+0,\ 0+6+0,\ 0+0+6)$

$\boxed{\text{grad}\,(\text{div}\, f) = (6,6,6)}$

$\text{curl}\, f = \nabla \times f$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix} = (-3x+3x)\hat{i} - (-3y+3y)\hat{j} + (-3z+3z)\hat{k} = (0,0,0)$$

$\text{grad}\,(\text{curl}\, f) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

(11) If $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$ be the fixed points and $P(x,y,z)$ be the variable point and $\overline{AP} = \bar{r}_1$, $\overline{BP} = \bar{r}_2$ then prove that

(i) $\nabla(\bar{r}_1\cdot\bar{r}_2) = \bar{r}_1 + \bar{r}_2$

(ii) $\nabla\cdot(\bar{r}_1\times\bar{r}_2) = 0$

(iii) $\nabla\times(\bar{r}_1\times\bar{r}_2) = 2(\bar{r}_1 - \bar{r}_2)$

→ Here → $\overline{AP} = \bar{r}_1 = (x-x_1, y-y_1, z-z_1)$

→ $\overline{BP} = \bar{r}_2 = (x-x_2, y-y_2, z-z_2)$

(i) $\nabla(\bar{r}_1\cdot\bar{r}_2) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\big((x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2)\big)$

$= \left(\frac{\partial}{\partial x}(x-x_1)(x-x_2);\ \frac{\partial}{\partial y}(y-y_1)(y-y_2),\ \frac{\partial}{\partial z}(z-z_1)(z-z_2)\right)$

$= \big((x-x_1)+(x-x_2),\ (y-y_1)+(y-y_2),\ (z-z_1)+(z-z_2)\big)$

$= \big((x-x_1),(y-y_1),(z-z_1)\big) + \big((x-x_2),(y-y_2),(z-z_2)\big)$

$= \bar{r}_1 + \bar{r}_2$

(ii) $\bar{r}_1\times\bar{r}_2 = (x-x_1, y-y_1, z-z_1)\times(x-x_2, y-y_2, z-z_2)$

$= \big((y-y_1)(z-z_2)-(z-z_1)(y-y_2),$
$(z-z_1)(x-x_2)-(x-x_1)(z-z_2),$
$(x-x_1)(y-y_2)-(y-y_1)(x-x_2)\big)$

$\nabla\cdot(\bar{r}_1\times\bar{r}_2) = \frac{\partial}{\partial x}\big((y-y_1)(z-z_2)-(z-z_1)(y-y_2)\big) +$
$\frac{\partial}{\partial y}\big((z-z_1)(x-x_2)-(x-x_1)(z-z_2)\big) +$
$\frac{\partial}{\partial z}\big((x-x_1)(y-y_2)-(y-y_1)(x-x_2)\big)$

$\therefore \nabla \cdot (\bar{r}_1 \times \bar{r}_2) = 0$

(iii) $\nabla \times (\bar{r}_1 \times \bar{r}_2)$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \Big((y-y_1)(z-z_2) - (z-z_1)(y-y_2), \\ (z-z_1)(x-x_2) - (x-x_1)(z-z_2), \\ (x-x_1)(y-y_2) - (y-y_1)(x-x_2) \Big)$$

$$= \Big(\frac{\partial}{\partial y}\left[(x-x_1)(y-y_2) - (y-y_1)(x-x_2)\right] - \frac{\partial}{\partial z}\left[(z-z_1)(x-x_2) - (x-x_1)(z-z_2)\right], \\ \frac{\partial}{\partial z}\left[(y-y_1)(z-z_2) - (z-z_1)(y-y_2)\right] - \frac{\partial}{\partial x}\left[(x-x_1)(y-y_2) - (y-y_1)(x-x_2)\right], \\ \frac{\partial}{\partial x}\left[(z-z_1)(x-x_2) - (x-x_1)(z-z_2)\right] - \frac{\partial}{\partial y}\left[(y-y_1)(z-z_2) - (z-z_1)(y-y_2)\right] \Big)$$

$$= \Big((x-x_1) - (x-x_2) - (x-x_2) + (x-x_1), \; (y-y_1) - (y-y_2) - (y-y_2) \\ + (y-y_1), \; (z-z_1) - (z-z_2) - (z-z_2) + (z-z_1) \Big)$$

$$= 2\Big((x-x_1) - (x-x_2), \; (y-y_1) - (y-y_2), \; (z-z_1) - (z-z_2) \Big)$$

$$= 2(\bar{r}_1 - \bar{r}_2)$$

$+ \Big(($

$+ \Big(($

UNIT 3

Ordinary point and singular point

Let $f(x,y)$ be a continuous function such that f_x, f_y exist
(1) A point on the curve $f(x,y)=0$ at which f_x and f_y do not vanish simultaneously is called ordinary point.
(2) A point of the curve at which f_x and f_y both vanish simultaneously is called a singular point. [A point of inflection which is a singular point.]

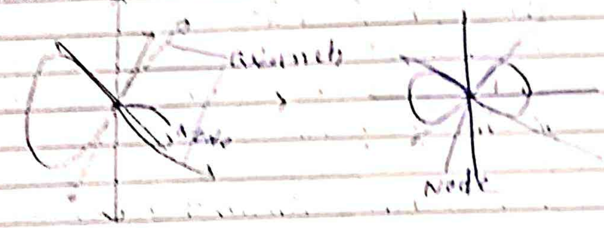
1. - Discuss Double points. OR
- Explain Double points of $f(x,y)=0$ OR
- Define Double point and prove the necessary condition for Double points of $f(x,y)=0$.

Ans. Double points:
A point on the curve $f(x,y)=0$ is called double point if two branches passes through that point.

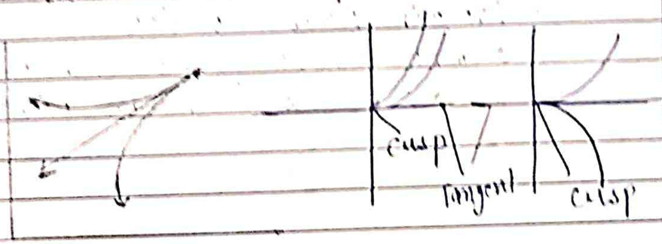
Now since there are two branches at a point so there must be two tangents, one to each branch.

These tangents are real and distinct, real and equal (coincident) OR imaginary depends upon slope of that tangents. So we divide Double points in three different types:

(1) Node : A double point on the curve at which two real and distinct tangents can be drawn is called Node.



(2) Cusp : A double point on the curve at which two real and equal (coincident) tangents can be drawn is called cusp.



(3) Isolated point : A double point on the curve at which two imaginary tangent can be drawn is said to be isolated point.

real tangent doesnot exist — isolated Point

A necessary condition for the existance of double point on the curve $f(x,y)=0$ is $f_x = 0$ and $f_y = 0$

Because $f(x,y) = 0$ ——(1)

$\Rightarrow d(f(x,y)) = 0$

$\Rightarrow f_x + f_y \frac{dy}{dx} = 0$ ——(2)

$\Rightarrow f_x + f_y \frac{dy}{dx}$

If $f_y \neq 0$, then the value of $\frac{dy}{dx}$ is unique, hence only one tangent can be drawn to the curve at point $P(x,y)$. Hence $P(x,y)$ is a double point on the curve, then two tangents must be drawn to the curve at P for which two values of dy/dx must be derived from eq^n (2) iff $f_x = 0$, $f_y = 0$. Thus the conditions that point $P(x,y)$ on the curve $f(x,y)=0$ is the double point are $f_x = 0$, $f_y = 0$

Again differentiate with respect to x, we get

$f_{xx} + f_{xy}\frac{dy}{dx} + \left(f_{yx} + f_{yy}\frac{dy}{dx}\right)\frac{dy}{dx} + f_y \frac{d^2y}{dx^2} = 0$

$\therefore f_{yy}\left(\frac{dy}{dx}\right)^2 + 2f_{xy}\left(\frac{dy}{dx}\right) + f_{xx} = 0 \quad (\because f_y = 0)$

Let $z = \frac{dy}{dx}$

$\Rightarrow f_{yy} z^2 + 2f_{xy} z + f_{xx} = 0$ ——(3)

$tz^2 + 2sz + r = 0$

$\Rightarrow \Delta = 4f_{xy}^2 - 4f_{yy}f_{xx}$ | $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$

$= 4(s^2 - rt)$

Case 1 : If $s^2 - rt > 0$ then the eq^n (3) has two real and distinct roots, which are the values of z ($= \frac{dy}{dx}$ = slope of tangent) so we have two real and distinct tangent and hence the point is called Node.

Some rect

$f(x,y)$

$\frac{d}{dx}(f(x,y))$

$\Rightarrow f_x + f_y \frac{dy}{dx}$

and $\frac{d}{dx}\left(f_y\left(\frac{dy}{dx}\right)\right)$

at point (x,y)

$f_{yx} + f_{yy}\frac{dy}{dx}\frac{dy}{dx}$

$+$

$f_y \frac{d^2y}{dx^2}$

0

double point (0,0) can be ... double point is other terms of $f(x,y)=0$. If double point ... origin should be shifted to that double point. ~~to apply this method~~

Case 2 : If $s^2-rt=0$, then the eqⁿ ⑤ has two real and equal roots, which are the values of z so we have two real and equal (coincident) tangents and hence the point is called cusp.

Case 3 : If $s^2-rt<0$, then the eqⁿ ⑤ has two imaginary roots, which are the values of z so we have two imaginary tangents and hence the point is called Isolated point.

For finding tangent follow page no. 86

(1) Find double point of $y^3 = x^3 + 3x^2$.

→ $f(x,y) = y^3 - x^3 - 3x^2 = 0$

$\Rightarrow f_x = -3x^2 - 6x = 0 \Rightarrow 3x(-x-2) = 0$

$\Rightarrow x = 0$ or $x = -2$

& $f_y = 3y^2 = 0 \Rightarrow y = 0$

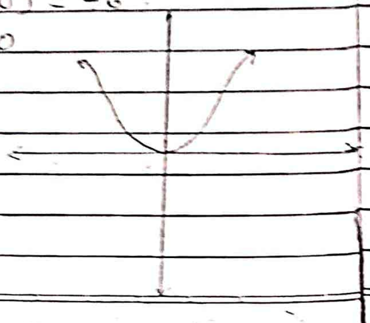
∴ we have $(0,0)$ & $(-2,0)$ which are the possible double points.

Now since $(0,0)$ is the only point on the curve so it is the only possible double point.

$f_{xx} = -6x - 6 \Rightarrow r = f_{xx}(0,0) = -6$

$f_{xy} = 0 \Rightarrow s = f_{xy}(0,0) = 0$

$f_{yy} = 6y \Rightarrow t = f_{yy}(0,0) = 0$



∴ $s^2 - rt = 0 - (-6)(0) = 0$

∴ $(0,0)$ is the cusp.

Let us equate the lowest degree term of the given curve to zero for getting tangent at the point.

$3x^2 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0, 0$

∴ $x=0$ line (i.e. y axis) is the tangent at the double point.

(2) Find double point of $x^3 + y^3 - 3axy = 0$.

→ $f(x,y) = x^3 + y^3 - 3axy = 0$

$\Rightarrow f_x = 3x^2 - 3ay = 0 \Rightarrow x^2 = ay$ —(1)

$f_y = 3y^2 - 3ax = 0 \Rightarrow y^2 = ax$ —(2)

From (1) & (2) we get,

$x^4 = a^2 ax$

$\Rightarrow x^4 = a^3 x$

$\Rightarrow x^4 - a^3 x = 0$

$\Rightarrow x(x^3 - a^3) = 0$

$\Rightarrow x = 0, y = 0$ or $x = a, y = a$

∴ We have $(0,0)$ & (a,a) which are the possible double points.

Now since $(0,0)$ is the only point on the curve so it is the only possible double point.

$f_{xx} = 6x \Rightarrow r = 0$

$f_{xy} = -3a \Rightarrow s = -3a$

$f_{yy} = 6y \Rightarrow t = 0$

∴ $s^2 - rt = 9a^2 > 0$

$\therefore$ (0,0) is the node.

Let us equate the lowest degree term of the given curve to zero for getting tangent at the point.

$-3axy = 0$

i.e. $x=0,\ y=0$

which are real and distinct tangents.

(3) Find double point of the curve

$y(y-6) = x^2(x-2)^3 - 9$

$\Rightarrow f(x,y) = y^2 - 6y - x^2(x-2)^3 + 9 = 0$

$\Rightarrow f_x = -2x(x-2)^3 + 3x^2(x-2)^2 = 0$

~~$\Rightarrow -x(x-2)^2(2x-4+3x) = 0$~~

$\Rightarrow -x(x-2)^2(5x-4) = 0$

$\Rightarrow x=0,\ x=2,\ x=2,\ x=4/5$

& $f_y = 2y - 6 = 0$

$\Rightarrow y = 3$

$\therefore$ We have $(0,3), (2,3), (4/5, 3)$ are the possible double points.

Also since (0,3) and (2,3) points on the curve they are possible double points.

$f_{xx} = -2(x-2)^3 - 6x(x-2)^2 - 6x(x-2)^2 + 6x^2(x-2)$

$\Rightarrow f_{xy} = 0$

$f_{yy} = 2$

(i) For point (0,3)

$r = -2(-2)^3 = +16,\ s = 0,\ t = 2$

$\Rightarrow s^2 - rt = 0 - 32 = -32 < 0$

$\therefore$ (0,3) is ~~node~~. isolated point.

For tangent line.

~~$y(y-6) = 0$~~

~~$y = 0\ ;\ y = 6$~~

~~which are real and different tangents.~~

(ii) For point (2,3)

$r = 2(0) + 6(2)(0) + 6(2)(0) + 6(0)(0)$

$= 0$

$s = 0,\ t = 2$

$\Rightarrow s^2 - rt = 0 - 0 = 0$

$\therefore$ (2,3) is cusp.

$x' = x - 2 \quad y' = y - 3$

$\therefore (0,0)$ is the node.

Let us equate the lowest degree term of the given curve to zero for getting tangent at the point.

$-3axy = 0$

i.e. $x=0,\ y=0$

which are real and distinct tangents.

(3) Find double point of the curve
$y(y-6) = x^2(x-2)^3 - 9$

$\Rightarrow f(x,y) = y^2 - 6y - x^2(x-2)^3 + 9 = 0$

$\Rightarrow f_x = -2x(x-2)^3 + 3x^2(x-2)^2 = 0$

$\Rightarrow -x(x-2)^2\ (2x-4+3x) = 0$

$\Rightarrow -x(x-2)^2(5x-4) = 0$

$\Rightarrow x=0,\ x=2,\ x=2,\ x=4/5$

& $f_y = 2y - 6 = 0$

$\Rightarrow y = 3$

$\therefore$ we have $(0,3), (2,3), (4/5, 3)$ are the possible double points.

Now since $(0,3)$ and $(2,3)$ points on the curve they are possible double points.

$f_{xx} = -2(x-2)^3 - 6x(x-2)^2 - 6x(x-2)^2 + 6x^2(x-2)$

$\Rightarrow f_{xy} = 0$

$f_{yy} = 2$

(i) For point $(0,3)$

$r = -2(-2)^3 = +16,\ s=0,\ t=2$

$\Rightarrow s^2 - rt = 0 - 32 = -32 < 0$

$\therefore (0,3)$ is ~~node~~ isolated point.

For tangent line.

$y(y-6) = 0$

$y=0\ ;\ y=6$

which are real and different tangents.

(ii) For point $(2,3)$

$r = 2(0) + 6(2)(0) + 6(2)(0) + 6(0)(0)$

$= 0$

$s = 0,\ t = 2$

$\Rightarrow s^2 - rt = 0 - 0 = 0$

$\therefore (2,3)$ is cusp.

$x' = x-2,\quad y' = y-3$

(Q) Find double points and their Nature.

(4) $(y-2)^2 = x(x-1)^2$

$\Rightarrow f(x,y) = (y-2)^2 - x(x-1)^2$

$f_x = -(x-1)^2 - 2x(x-1) = 0$ — (1)

$f_y = 2(y-2) = 0 \Rightarrow y = 2$

From eq^n (1) $(x-1)(-x+1-2x) = 0$

$\Rightarrow (x-1)(-3x+1) = 0$

$\Rightarrow x = 1$ or $x = \frac{1}{3}$

$\therefore (1,2)$ and $(\frac{1}{3}, 2)$ are possible double points

but $(1,2)$ is only point on the curve. So it is only possible double point.

$f_{xx} = -2(x-1) - 2(x-1) - 2x \Rightarrow f_{xx} = -4(x-1) - 2x$

$f_{xy} = 0$

$f_{yy} = 2$

$\therefore r = -4(1-1) - 2(1) = -2$

$s = 0$

$t = 2$

$\therefore s^2 - rt = 4 > 0$

$\therefore (1,2)$ is node. [For tangent eq^{n}s. see p.no. 86]

(5) $x^3 + y^3 - 3xy = 0$.

$\Rightarrow f(x,y) = x^3 + y^3 - 3xy = 0$

$f_x = 3x^2 - 3y = 0 \Rightarrow x^2 = y$ — (1)

$f_y = 3y^2 - 3x = 0 \Rightarrow y^2 = x$ — (2)

From eq^n (1) & (2) we get $y^4 = y \Rightarrow y(y^3 - 1) = 0$

$\Rightarrow y = 0$ or $y = 1$

If $y = 0 \Rightarrow x = 0$ & $y = 1 \Rightarrow x = 1$

$\therefore (0,0)$ & $(1,1)$ are possible double points.

Now since $(0,0)$ is only point on the curve. So it is only possible double point.

$f_{xx} = 6x$, $f_{xy} = -3$, $f_{yy} = 6y$

$\therefore r = 0, s = -3, t = 0$

$\therefore s^2 - rt = 9 > 0$

$\therefore (0,0)$ is node.

For tangent eq^n at $(0,0)$, ~~compare~~ equate lowest degree with 0.

$\therefore 3xy = 0 \Rightarrow x = 0$ & $y = 0$ are tangent lines.

(6) $x^3(x+y) = y^2$

$\Rightarrow f(x,y) = x^3(x+y) - y^2$

$f_x = 3x^2(x+y) + x^3 = 0$ — (1)

$f_y = x^3 - 2y = 0$ — (2)

From (1) & (2), we get

$3x^2(x+y) + x^3 = x^3 - 2y$

$\Rightarrow 3x^3 + 3x^2y + 2y = 0$

$\Rightarrow 3x^3 + 3x^2\frac{x^3}{2} + 2\frac{x^3}{2} = 0$ $(\because y = \frac{x^3}{2}$ eq^n 2)

$\Rightarrow 3x^3 + \frac{3}{2}x^5 + x^3 = 0$

$\Rightarrow \frac{3}{2}x^5 + 4x^3 = 0$

$\Rightarrow 3x^5 + 8x^3 = 0$

$\Rightarrow x^3(3x^2 + 8) = 0$

$\Rightarrow x = 0 \; ; \; 3x^2 + 8 \neq 0$

$\Rightarrow y = 0$

$\therefore (0,0)$ is possible double point.

$f_{xx} = 6x(x+y) + 3x^2 + 3x^2$

$f_{xy} = 3x^2$

$f_{yy} = -2$

$\therefore r = 0, s = 0, t = -2$

$\therefore s^2 - rt = 0$

$\therefore (0,0)$ is cusp.

For tangent eqn

$y^2 = 0 \Rightarrow y = 0, y = 0$ are tangent eqns.

$(y-2)^2 = x(x-1)^2$

$f(x,y) = (y-2)^2 - x(x-1)^2$

$f_x = -(x-1)^2 - 2x(x-1) = 0 \quad —(1)$

$f_y = 2(y-2) = 0$

$\therefore y = 2$

From eqn (1)

$-(x-1)^2 - 2x(x-1) = 0$

$\therefore (x-1)(-x+1-2x) = 0$

$\therefore (x-1)(1-3x) = 0$

$\therefore x = 1 \qquad x = \frac{1}{3}$

$\therefore (1,2), (\frac{1}{3}, 2)$ are possible double points.

but $(1,2)$ only on the curve.

$f_{xx} = -2(x-1) - 2(x-1) - 2x$

$= -4(x-1) - 2x$

$f_{xy} = 0$

$f_{yy} = 2$

$r = f_{xx}(1,2) = -2$

$s = f_{xy}(1,2) = 0$

$t = f_{yy}(1,2) = 2$

$\therefore s^2 - rt = 0 - (-2)(2) = 4 > 0$

$\therefore (1,2)$ is node.

For finding tangent equations.

Ref. (P.No. 47)

$$\frac{dy}{dx} = \frac{-s \pm \sqrt{4s^2 - 4rt}}{2t}$$

$$= \frac{-0 \pm \sqrt{4(0) - 4(-2)(2)}}{2(2)}$$

$$= \frac{\pm 4}{4}$$

$$= \pm 1$$

$$\frac{dy}{dx} = \frac{-2s \pm \sqrt{4(s^2 - rt)}}{2t}$$

$$= \frac{-2s \pm 2\sqrt{s^2 - rt}}{2t}$$

$$= \frac{-s \pm \sqrt{s^2 - rt}}{t}$$

eq^n of line passing through $(1,2)$ with slope $m = 1$ is

$y - 2 = 1(x-1)$

$\Rightarrow y - 2 = x - 1$

$\Rightarrow y = x + 1$

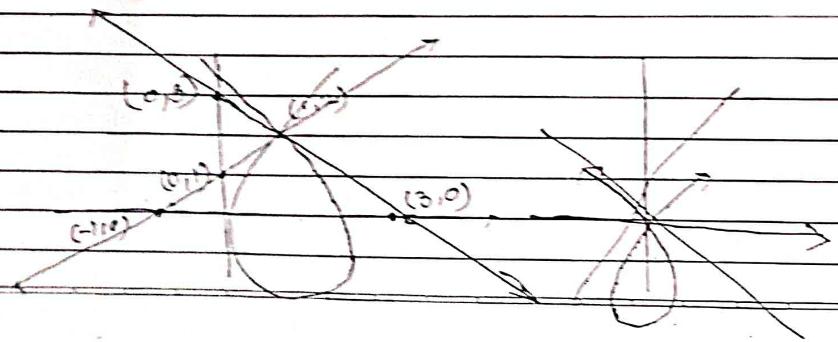
eq^n of line passing through $(1,2)$ with slope $m = -1$ is

$y - 2 = -1(x-1)$

$\Rightarrow y - 2 = -x + 1$

$\Rightarrow y = -x + 3$

$\therefore y = x+1$ & $y = -x+3$ are tangent eq^{n}s.



★ Ordinary Point :

A point which lies on at least one ordinary line is called an ordinary point, or sometimes a regular point.

★ Ordinary line :

Given an arrangement of points, a line containing just two of them is called an ordinary line.