B Sc Sem 3

Paper 303

UNIT 1

Limits And Continuity

Limit of a function in two variables :

Let $f(x, y)$ be a well defined function in the deleted δ-neighbourhood of a point (a, b), Let $l \in$ R.

Now, if for each $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|(x,y) - (a,b)| < \delta \quad \Rightarrow \quad |f(x,y) - l| < \varepsilon,$$

then we say that $\lim\limits_{(x,y)\to(a,b)} f(x, y) = l.$

Limit of a function in n variables :

Let $f(x_1, x_2, x_3, \ldots, x_n)$ be a well defined function in the deleted δ-neighbourhood of a point $(a_1, a_2, \ldots, a_n)$. Let $l \in$ R.

Now , if for each $\varepsilon > 0$, $\exists\ \delta > 0$ Such that

$$|(x_1, x_2, x_3, \ldots x_n) - (a_1, a_2, \ldots, a_n)| < \delta \quad \Rightarrow \quad |f(x_1, x_2, x_3, \ldots x_n) - l| < \varepsilon,$$

then we say that $\lim\limits_{(x_1, x_2, \ldots, x_n)\to(a_1, a_2, \ldots, a_n)} f(x_1, x_2, x_3, \ldots x_n) = l.$

Examples

1) Using the definition of a limit, prove that $\lim\limits_{(x,y)\to(2,3)} 3xy = 18$

By the definition of limit, we want to prove $\lim\limits_{(x,y)\to(2,3)} 3xy = 18$

Let $\varepsilon > 0$.

Since $(x, y) \to (2, 3)$, suppose there is $\delta > 0$

such that $|(x, y) - (2, 3)| < \delta$.

$\Rightarrow \quad |x - 2| < \delta \quad$ and $\quad |y - 3| < \delta$

$\Rightarrow \quad -\delta < x - 2 < \delta \quad$ and $-\delta < y - 3 < \delta$

$\Rightarrow \quad 2 - \delta < x < 2 + \delta \qquad \ldots(1)$ and

$3 - \delta < y < 3 + \delta \qquad \ldots(2)$

From(1) and (2),

$\Rightarrow (2 - \delta)(3 - \delta) \quad < \quad xy \quad < \quad (2 + \delta)(3 + \delta)$

$\Rightarrow 6 - 2\delta - 3\delta + \delta^2 \quad < \quad xy \quad < \quad 6 + 2\delta + 3\delta + \delta^2$

$\Rightarrow \delta^2 - 5\delta + 6 \quad < \quad xy \quad < \quad \delta^2 + 5\delta + 6$

$\Rightarrow 3\delta^2 - 15\delta + 18 \quad < \quad 3xy \quad < \quad 3\delta^2 + 15\delta + 18$

$\Rightarrow 3\delta^2 - 15\delta \quad < \quad 3xy - 18 \quad < \quad 3\delta^2 + 15\delta$

$\Rightarrow -3\delta^2 - 15\delta \quad < \quad 3xy - 18 \quad < \quad 3\delta^2 + 15\delta$

$(\because -3\delta^2 - 15\delta < 3\delta^2 + 15\delta)$

$\therefore |3xy - 18| < 3\delta^2 + 15\delta$

If we take $3\delta^2 + 15\delta = \varepsilon$ thcn $|3xy - 18| < \varepsilon$

Also, the equation becomes,

$3\delta^2 + 15\delta - \varepsilon = 0$

This is a quadratic equation in δ with $a = 3, b = 15, c = -\varepsilon$

$$\delta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-15 + \sqrt{225 - 4(3)(-\varepsilon)}}{6}$$

$$= \frac{-15 + \sqrt{225 - 4(3)(-\varepsilon)}}{6} \quad > \quad 0$$

$(\because \sqrt{225+12\varepsilon} > 15)$

$\therefore$ Taking $3\delta^2 + 15\delta = \varepsilon$,

$|3xy-18| < \varepsilon$

Thus, for any $\varepsilon > 0$ we have $\delta > 0$ such that

$|(x,y)-(2,3)| < \delta \Rightarrow |3xy-18| < \varepsilon$

$\therefore \lim_{(x,y)\to(2,3)} 3xy = 18$

2) $\lim_{(x,y)\to(2,5)} (x^2 + 2y) = 14$

By the definition of limit, we want to prove $\lim_{(x,y)\to(2,5)} (x^2 + 2y) = 14$

Let $\varepsilon > 0$.

Since $(x, y) \to (2, 5)$, suppose there is $\delta > 0$ such that

$|(x, y) - (2, 5)| < \delta$

$\Rightarrow |x-2| < \delta$ and $|y-5| < \delta$

$\Rightarrow -\delta < x-2 < \delta$ and $-\delta < y-5 < \delta$

$\Rightarrow 2-\delta < x < 2+\delta$ ...(1) and

$\Rightarrow 5-\delta < y < 5+\delta$ ...(2)

From (1) we get, $(2-\delta)^2 < x^2 < (2+\delta)^2$

$4-4\delta+\delta^2 < x^2 < 4+4\delta+\delta^2$ ...(3)

From (2) we get,

$2(5-\delta) < 2y < 2(5+\delta)$

$10-2\delta < 2y < 10+2\delta$ ...(4)

From (3) and (4),

$4-4\delta+\delta^2+10-2\delta < x^2+2y < 4+4\delta+\delta^2+10+2\delta$

$\therefore \delta^2-6\delta+14 < x^2+2y < \delta^2+6\delta+14$

$\therefore \delta^2-6\delta < x^2+2y-14 < \delta^2+6\delta$

$\therefore -\delta^2-6\delta < x^2+2y-14 < \delta^2+6\delta$

$(\because -\delta^2-6\delta < \delta^2-6\delta)$ $\therefore$

$|x^2+2y-14| < \delta^2+6\delta$

If we take $\delta^2+6\delta = \varepsilon$

Then equation becomes

$\delta^2 + 6\delta - \varepsilon = 0$

$\therefore \delta = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$

$= \dfrac{-6+\sqrt{36-4(1)(-\varepsilon)}}{2}$ =

$\dfrac{-6+\sqrt{36+4\varepsilon}}{2} > 0$

$\because$ Taking $\delta^2 + 6\delta = \varepsilon$

$|x^2 + 2y - 14| < \varepsilon$

Thus, for any $\varepsilon > 0$, we have $\delta > 0$ such that

$|(x, y) - (2, 5)| < \delta \Rightarrow |x^2 + 2y - 14| < \varepsilon$

$\therefore \lim\limits_{(x,y)\to(2,5)} (x^2 + 2y) = 14$

3) $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^2+y^2}{x+y} = 1$

By the definition of limit, we want to prove $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^2+y^2}{x+y} = 1$

Let $\varepsilon > 0$

Since $(x, y)\to(1, 1)$, suppose there is $\delta > 0$ such that

$|(x, y) - (1, 1)| < \delta$

$\Rightarrow \quad |x-1| < \delta$ and $|y-1| < \delta$

$\Rightarrow \quad -\delta < x-1 < \delta$ and $|y-1| < \delta$

$\Rightarrow \quad 1-\delta < x < 1+\delta \quad \ldots.(1)$

$\Rightarrow \quad 1-\delta < y < 1+\delta \quad \ldots.(2)$

Now, we want to show that

$\left|\dfrac{x^2+y^2}{x+y} - 1\right| < \varepsilon$

$\therefore \left|\dfrac{(x^2+y^2)-(x+y)}{x+y}\right| < \varepsilon$

$\therefore$ Now from (1),

$1 - 2\delta + \delta^2 < x^2 < 1 + 2\delta + \delta^2 \quad \ldots.(3)$

From (2),

$1 - 2\delta + \delta^2 < y^2 < 1 + 2\delta + \delta^2 \quad \ldots.(4)$

From (3) and (4),

$1 - 2\delta + \delta^2 + 1 - 2\delta + \delta^2 < x^2 + y^2 < 1 + 2\delta + \delta^2 + 1 + 2\delta + \delta^2$

$\therefore 2\delta^2 - 4\delta + 2 < x^2 + y^2 < 2\delta^2 + 4\delta + 2 \quad \ldots.(5)$

From (1) and (2),
$(1-\delta)+(1-\delta) < x+y < (1+\delta)+(1+\delta)$

$\therefore \quad 1-\delta+1-\delta < x+y < 1+\delta+1+\delta$

$\therefore \quad 2-2\delta < x+y < 2+2\delta$

$\therefore \quad -(2-2\delta) > -(x+y) > -(2+2\delta)$

$\therefore \quad -2-2\delta < -(x+y) < -2+2\delta \quad \ldots.(6)$

From (5) and (6),
$(2\delta^2-4\delta+2)+2-2-2\delta < (x^2+y^2)-(x+y) < (2\delta^2-4\delta+2)+(-2+2\delta)$

$\therefore \quad 2\delta^2-4\delta+2+2-2-2\delta < (x^2+y^2)-(x+y) < 2\delta^2-4\delta+2-2+2\delta$

$\therefore \quad 2\delta^2-6\delta < (x^2+y^2)-(x+y) < 2\delta^2+6\delta \quad \ldots.(7)$

Now we have,

$2-2\delta < x+y < 2+2\delta$

$$\therefore \frac{1}{2+2\delta} < \frac{1}{x+y} < \frac{1}{2-2\delta} \quad \ldots.(8)$$

From (7) and (8),

$$\frac{2\delta^2-6\delta}{2+2\delta} < \frac{(x^2+y^2)-(x+y)}{x+y} < \frac{2\delta^2+6\delta}{2-2\delta}$$

$$\therefore \frac{\delta^2-3\delta}{1+\delta} < \frac{(x^2+y^2)-(x+y)}{x+y} < \frac{\delta^2+3\delta}{1-\delta}$$

Now $1+\delta > 1-\delta \Rightarrow \frac{1}{1+\delta} < \frac{1}{1-\delta} \Rightarrow$

$$\frac{\delta^2+3\delta}{1+\delta} < \frac{\delta^2+3\delta}{1-\delta} \Rightarrow \frac{-(\delta^2+3\delta)}{1+\delta} > \frac{-(\delta^2+3\delta)}{1-\delta}$$

$$\therefore \frac{-(\delta^2+3\delta)}{1-\delta} < \frac{(x^2+y^2)-(x+y)}{x+y} < \frac{\delta^2+3\delta}{1-\delta}$$

$$\therefore \left|\frac{(x^2+y^2)-(x+y)}{x+y}\right| < \frac{\delta^2+3\delta}{1-\delta}$$

If we take $\frac{\delta^2+3\delta}{1-\delta} = \varepsilon$

Then equation becomes,

$$\delta^2 + 3\delta = \varepsilon - \delta\varepsilon$$

$$\therefore \quad \delta^2 + 3\delta - \varepsilon + \delta\varepsilon = 0$$

$$\therefore \quad \delta^2 + (3+\varepsilon)\delta - \varepsilon = 0$$

$$\therefore \quad \delta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(3+\varepsilon) + \sqrt{(3+\varepsilon)^2 - 4(1)(-\varepsilon)}}{2}$$

$$= \frac{-(3+\varepsilon) + \sqrt{(3+\varepsilon)^2 + 4\varepsilon}}{2} \quad > \quad 0$$

$$(\because \sqrt{(3+\varepsilon)^2 + 4\varepsilon} > (3+\varepsilon))$$

$$\therefore \text{ Taking } \frac{\delta^2 + 3\delta}{1-\delta} = \varepsilon$$

$$\left| \frac{(x^2+y^2)}{(x+y)} - 1 \right| < \varepsilon$$

Thus, for any $\varepsilon > 0$, we have $\delta > 0$ such that $|(x - y) - (1, 1)| < \delta$
$\Rightarrow \quad |f(x, y) - 1| < \varepsilon$
Thus,

$$\lim_{(x,y)\to(1,1)} \frac{x^2+y^2}{x+y} = 1$$

4) $$\lim_{(x,y)\to(2,1)} \frac{2x+y}{3y-x}$$

$$\left[\frac{21\delta}{1-4\delta} = \varepsilon \right]$$

5) $$\lim_{(x,y)\to(2,3)} x^2 + xy = 10$$

$$[\, 2\,\delta^2 + 9\,\delta = \varepsilon \,]$$

6) $\lim\limits_{(x,y)\to(4,-1)} 3x-2y=14$

$[5\,\delta=\varepsilon]$

7) $\lim\limits_{(x,y)\to(1,2)} 3xy=6$

$[3\delta^2+9\,\delta=\varepsilon\,]$

8) $\lim\limits_{(x,y)\to(1,2)} x^2+2y=5$

$[\delta^2+4\,\delta\;=\;\varepsilon]$

9) $\lim\limits_{(x,y)\to(2,-3)} 9x-2y=24$

$[11\delta=\;\varepsilon]$

10) $\lim\limits_{(x,y)\to(2,1)} \dfrac{2x^2+y}{x+4y}=\dfrac{3}{2}$

$$\left[\frac{4\delta^2+29\delta}{6+5\delta}=\varepsilon\right]$$

11) $\lim\limits_{(x,y)\to(3,2)} x^2y+xy^2=30$

$[2\delta^3+15\delta^2+37\delta=\varepsilon]$

12) $\lim\limits_{(x,y)\to(1,2)} \dfrac{2x+3y}{3x+y}=\dfrac{8}{5}$

$$\left[\frac{21\delta}{25-20\delta}=\varepsilon\right]$$

Iterated Limits

Let f(x, y) be a well defined function in the neighborhood of a point (a, b). Then, if

$$\lim_{x\to a}\left\{\lim_{y\to b} f(x,y)\right\} \quad \text{and} \quad \lim_{y\to b}\left\{\lim_{x\to a} f(x,y)\right\}$$

Exist, then they are called iterated limits.

These limits may or may not be equal, but whether they are equal or not equal, $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ may or may not exist.

However, if $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists, iterated limits must be equal. Now we see an example where iterated limits are not equal.

1) Let f(x, y)= $\begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$

$$\lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{x^2}{x^2}\right\}$$

$$= \lim_{x\to 0}(1)$$

= 1.

$$\lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x^2-y^2}{x^2+y^2}\right\}$$

$$= \lim_{y\to 0}\left\{\frac{-y^2}{y^2}\right\}$$

$$= \lim_{y\to 0}(-1)$$

= -1

Note :- In above example, iterated limits are not equal. So,

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

2) Prove that $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x-y}{x+y}$ does not exist.

Solution :-

Here, we want to find iterated limits.

$$\lim_{x\to 0}\left\{\lim_{y\to 0}\frac{2x-y}{x+y}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{2x}{x}\right\}$$

$$= \lim_{x\to 0}(2)$$

= 2

$$\lim_{y\to 0}\left\{\lim_{x\to 0}\frac{2x-y}{x+y}\right\}$$

$$= \lim_{y\to 0}\left\{\frac{-y}{y}\right\}$$

$$= \lim_{y\to 0}(-1)$$

$= -1$

Thus, $\lim_{x\to 0}\left\{\lim_{y\to 0}\frac{2x-y}{x+y}\right\} \neq \lim_{y\to 0}\left\{\lim_{x\to 0}\frac{2x-y}{x+y}\right\}$

$\therefore \lim_{(x,y)\to(0,0)} \frac{2x-y}{x+y}$ does not exist.

Note :- We derived that if iterated limits are not equal, then

$\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

There are example where iterated limits are equal but

$\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

$\rightarrow$

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Thus, $\lim_{x\to 0} f(x,\phi_1(x)) \neq \lim_{x\to 0} f(x,\phi_2(x))$

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

○ Note :- we have following obvious facts :

1. $\lim_{(x,y)\to(a,b)} (f(x,y)+g(x,y))$

$= \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$

2. $\lim_{(x,y)\to(a,b)} (f(x,y)\times g(x,y))$

$= \lim_{(x,y)\to(a,b)} f(x,y) \times \lim_{(x,y)\to(a,b)} g(x,y)$

3. $\lim_{(x,y)\to(a,b)} (f(x,y)\div g(x,y))$

$= \lim_{(x,y)\to(a,b)} f(x,y) \div \lim_{(x,y)\to(a,b)} g(x,y)$

<u>Definition</u>:-

Let $f(x, y)$ be a real function defined on a field R^2. Let $(a,b) \in R^2$.

then f is said to be a continuous function at point (a,b)

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

○ Note : There are many simple example of continuous functions.

For example :-

1) Let $f(x,y) = x^2 + y^2$, then $f(x,y)$ is continuous at all points (a,b) because,

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} x^2 + y^2$$

$$= \lim_{(x,y)\to(a,b)} a^2 + b^2 = f(a,b)$$

However, there are many discontinuous function also.

For example:-

2) Let $f(x,y) = \begin{cases} x^2 + y^2 & if\ (x,y) \neq (2,3) \\ x^2 - y^2 & if\ (x,y) = (2,3) \end{cases}$ then

$$\lim_{(x,y)\to(2,3)} f(x,y) = \lim_{(x,y)\to(2,3)} x^2 + y^2 = 13$$

But $f(2,3) = (2)^2 - (3)^2 = -5$

Thus $\lim_{(x,y)\to(a,b)} f(x,y) \neq f(2,3)$. Therefore $f(x,y)$ is not continuous at

$(2,3)$.

Continuity in n-variables :- Let $f(x_1, x_2, x_3, ..., x_n)$ be a real valued function defined on a field R^n. Let $f(a_1, a_2, a_3, ..., a_n) \in R^n$ .Then f is said

to be a continuous function at point $f(a_1, a_2, a_3, ..., a_n)$.

If $\lim_{(x_1,x_2,x_3,....,x_n)\to(a_1,a_2,a_3,....,a_n)} f(x_1, x_2, x_3, ..., x_n) = f(a_1, a_2, a_3, ..., a_n)$

Example 1: **[2008]**

Discuss continuity of the

$$f(x, y) = \begin{cases} \dfrac{x^2 y^3}{x^5 + y^5}, (x, y) \neq (0, 0) \\ 1 \qquad ,(x, y) = (0, 0) \end{cases}$$

Solution :- Here $f(x, y) = \dfrac{x^2 y^3}{x^5 + y^5}$ is homogenous because total power of

all terms of numerator and denominator have equal powers in such cases,

take $y = mx = \phi(x)$

take ,then,

so $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \dfrac{x^2 y^3}{x^5 + y^5} = \lim_{x\to 0} \dfrac{x^2 m^3 x^3}{x^5 + m^5 x^5}$

$= \lim_{x\to 0} \dfrac{m^3}{1+m^5}$

Here for different values of m, above limit is different.

For $m = 2$, limit is $\dfrac{(2)^3}{1+(2)^5}$ and for $m = 3$ limit is $\dfrac{(3)^3}{1+(3)^5}$

So by theorem, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. Thus function is not

Continuous at $(0, 0)$

Example 2: Discuss the continuity of $f(x, y, z) = x^2 + y^2 + z^2 - xyz$

Solution :- Let $(a, b, c) \in R^3$,

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = \lim_{(x,y,z)\to(a,b,c)} x^2+y^2+z^2-xyz$$

$= a^2+b^2+c^2-abc$ ....(1)

Also, $f(a,b,c) = a^2+b^2+c^2-abc$ ....(2)

from (1) and (2),

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$$

$\therefore f$ is continuous at point (a,b,c)

Thus $f(x,y,z)$ is continuous at all points of R^3 .i.e.

f is continuous on R^3.

Note:- if f and g are continuous functions, then their composition is also a continuous function.

For example:

If $f(x) = \cos x$ and $g(x,y) = x^2+y^2$ and their composition function

$\cos(x^2+y^2)$ is also a continuous function.

<u>Example 3</u>: **[2012]**

Let $f(x,y) = \dfrac{x^2y^4}{(x^2+y^4)^2}$ if $(x,y)\neq(0,0)$ and $f(0,0)=0$ then discuss continuity of f at $(0,0)$.

<u>Solution</u>:-

Take $y=\sqrt{mx}=\phi(x)$ then $\phi(x)$ is continuous and $\phi(0)=0$

$$\therefore \lim_{(x,y)\to(0.0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y^4}{(x^2+y^4)^2}$$

$$= \lim_{x\to 0} \frac{x^2m^2x^2}{(x^2+m^2x^2)^2}$$

$$=\lim_{x\to 0}\frac{m^2x^4}{x^4(1+m^2)^2}$$

$$=\lim_{x\to 0}\frac{m^2}{(1+m^2)^2}$$

Here, for different values of m, above limit is different.

So by theorem, $\lim_{(x,y)\to(0.0)} f(x,y)$ does not exist.

So $f(x,y)$ is not continuous at $(0,0)$

Example 4:**[2012]**

$f(x,y)=\dfrac{2xy^2}{x^3+y^3}$ if $(x,y)\neq(0,0)$ and $f(0,0)=0$ then discuss continuity at $(0,0)$.

Solution:- Let $y=mx=\phi(x)$ then $\phi(x)$ is continuous and $\phi(0)=0$

$$=\lim_{x\to 0}\frac{2xm^2x^2}{x^3+m^3x^3}$$

$$=\lim_{x\to 0}\frac{2m^2}{1+m^3}=\frac{2m^2}{1+m^3}$$

Here, for different values of m, above limit is different

So by theorem,

$\lim_{(x,y)\to(0.0)} f(x,y)$ does not exist.

So, $f(x,y)$ is not continuous at $(0,0)$

Example 5:

$f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$ if $(x,y)\neq(0,0)$ and $f(0,0)=0$ then discuss continuity at $(0,0)$.

Solution:

Let $y = mx = \phi(x)$ then, $\phi(x)$ is continuous and $\phi(x) = 0$

$$\therefore \lim_{(x,y)\to(0.0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

$$= \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2}$$

$$= \lim_{x\to 0} \frac{x^2(1-m^2)}{x^2(1+m^2)}$$

$$= \frac{(1-m^2)}{(1+m^2)}$$

Here for different values of m, above limit is different.

So by the theorem.

$\lim_{(x,y)\to(0.0)} f(x,y)$ does not exist

$f(x,y)$ is not continuous at $(0,0)$

Unit 2
Partial Differentiation

○ **Definition**:- Let $f(x,y)$ be defined on a set $S \subset R^2$. Let $(a,b) \in R^2$,

Then the partial derivative of $f(x,y)$ at point (a,b) is denoted and defined as

$$\left(\frac{\partial f}{\partial x}\right) = f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Similarly, we define,

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

Note:- from the above definition , we get ,

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

and $f_y(0,0) = \lim_{k \to 0} \dfrac{f(0,k) - f(0,0)}{k}$

<u>Examples (without using definition)</u>

(1) If $f(x,y) = x^3 + y^3 + 3x^2y^2$, then find f_x and f_y.

<u>Solution</u>: $f(x,y) = x^3 + y^3 + 3x^2y^2$

$\therefore f_x = 3x^2 + 6xy^2$ and $f_y = 3y^2 + 6x^2y$

(2) If $f(x,y) = \log(x^2 + y^2)$, then find f_x and f_y.

<u>Solution</u>: $f(x,y) = \log(x^2 + y^2)$

$$f_x = \frac{1}{x^2+y^2}\cdot 2x = \frac{2x}{x^2+y^2} \quad \text{and} \quad f_y = \frac{1}{x^2+y^2}\cdot 2y = \frac{2y}{x^2+y^2}$$

(3) $f(x,y) = \tan^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, then prove that $\frac{\partial f}{\partial x} = -\frac{y}{x}\frac{\partial f}{\partial y}$

Solution: (Self Try)

(4) $f(x,y) = \begin{cases} \frac{x(x^2-y^2)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0 & , (x,y)=(0,0) \end{cases}$ **[2012]**

Then find $f_x(0,0)$ and $f_y(0,0)$.

[Note:- To find $f_x = (0,0)$ and $f_y = (0,0)$, we normally find f_x first and then put $x = 0$, $y = 0$ in it . But in this examples, if we do so, then $f_x = (0,0) = \frac{0}{0}$ comes. So we cannot follow that method.so we use the definition of partial derivatives.]

We have,

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0}\frac{f\,\frac{h(h^2-0)}{h^2+0}}{h}$$

$$= \lim_{h\to 0}(1) = 1$$

Similarly , $f_y(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = 0$

(5) Find $f_x(0,0)$ and $f_y(0,0)$ for $f(x,y) = \begin{cases} \frac{y^3-x^3}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0 & , (x,y)=(0,0) \end{cases}$

$$f(x,y)=\frac{y^3-x^3}{y^2+x^2}$$

$$\therefore f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{0-h^3}{h^2}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{-h^3}{h^2}}{h}$$

$$=\lim_{h\to 0}(-1)=-1$$

$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}$$

$$=\lim_{k\to 0}\frac{\frac{k^3-0-0}{k^2+0}}{k}$$

$$=1$$

(6) $$f(x,y)=\begin{cases}\frac{x\,y}{x^2+y^2} & ,(x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$$

Show that $f_x(0,0)$ and $f_y(0,0)$ exist but f is not continuous at $(0,0)$.

$$f(x,y)=\frac{x\,y}{x^2+y^2}$$

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{(h)(0)-0}{h^2+0}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{0}{h^2}}{h}$$

$$= 0$$

Similarly , $f_y(0,0) = \lim_{k\to 0} \frac{(0,k) - f(0,0)}{k}$

$$= 0$$

Let $y = mx = \phi(x)$ then $\phi(x)$ is continuous and $\phi(0) = 0$

So $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x\,y}{x^2 + y^2}$

$$= \lim_{x\to 0} \frac{mx^2}{x^2 + m^2x^2}$$

$$= \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)}$$

$$= \lim_{x\to 0} \frac{m}{1+m^2}$$

Here, for different values of m, above limit is different. So, by the theorem,

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\therefore f$ is not continuous at $(0,0)$.

(7) If $f(x,y) = \begin{cases} \frac{(x^2y - xy^2)}{x^2 + y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$

Then discuss the continuity of f_x and f_y at $(0,0)$.

Here $f_x = \frac{(x^2 + y^2)(2xy - y^2) - (x^2y - xy^2)(2x)}{(x^2 + y^2)^2}$

$$= \frac{2x^3y + 2xy^3 - x^2y^2 - y^4 - 2x^3y + 2x^2y^2}{x^4 + 2x^2y^2 + y^4}$$

$$= \frac{2xy^3 + x^2y^2 - y^4}{x^4 + 2x^2y^2 + y^4}$$

Thus $f_x(x, y) = \begin{cases} \dfrac{2xy^3 + x^2y^2 - y^4}{x^4 + 2x^2y^2 + y^4} & ,(x, y) \neq (0,0) \\ 0 & ,(x, y) = (0,0) \end{cases}$

$$\therefore \lim_{(x,y)\to(0,0)} f_x(x, y) = \lim_{(x,y)\to(0,0)} \frac{2xy^3 + x^2y^2 - y^4}{x^4 + 2x^2y^2 + y^4}$$

Taking $y = mx = \phi(x)$, $\phi(x)$ is continuous and $\phi(0) = 0$.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{2xy^3 + x^2y^2 - y^4}{x^4 + 2x^2y^2 + y^4} = \lim_{x\to 0} \frac{2xm^3x^3 + x^2m^2x^2 - m^4x^4}{x^4 + 2x^2m^2x^2 + m^4x^4}$$

$$= \lim_{x\to 0} \frac{x^4(2m^3 + m^2 - m^4)}{x^4(1 + 2m^2 + m^4)}$$

$$= \lim_{x\to 0} \frac{(2m^3 + m^2 - m^4)}{(1 + 2m^2 + m^4)}$$

$$= \frac{(2m^3 + m^2 - m^4)}{(1 + 2m^2 + m^4)}$$

Here, for different values of m, above limit is different.

So by the theorem $\lim_{(x,y)\to(0,0)} f_x(x, y)$ does not exist.

So , f_x is not continuous at $(0,0)$.

Now,

$$f_y = \frac{(x^2+y^2)(x^2-2xy)-(x^2y-xy^2)(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^4+x^2y^2-2x^3y-2x^2y^2+2xy^3-2xy^3}{x^4+2x^2y^2+x^4}$$

$$= \frac{x^4-x^2y^2-2x^3y}{x^4+2x^2y^2+y^4}$$

Thus, $f_y(x,y) \begin{cases} = \dfrac{x^4-x^2y^2-2x^3y}{x^4+2x^2y^2+y^4} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$

$$\therefore \lim_{(x,y)\to(0,0)} f_y(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4-x^2y^2-2x^3y}{x^4+2x^2y^2+y^4}$$

taking $y = mx = \phi(x)$ then $\phi(x)$ is continuous and $\phi(0)=0$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^4-x^2y^2-2x^3y}{x^4+2x^2y^2+y^4}$$

$$= \lim_{x\to 0} \frac{x^4-x^2m^2x^2-2x^3mx}{x^4+2x^2m^2x^2+m^4x^4}$$

$$= \lim_{x\to 0} \frac{x^4(1-m^2-2m)}{x^4(1+2m^2+m^4)}$$

$$= \lim_{x\to 0} \frac{(1-m^2-2m)}{(1+2m^2+m^4)}$$

$$= \frac{1-m^2-2m}{1+2m^2+m^4}$$

Here for different values of m , above limit is different.

So , by the theorem , $\lim_{(x,y)\to(0,0)} f_y(x,y)$ does not exist

So , f_y is not continuous at $(0,0)$.

(8) $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^3+y^3} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$

then show that $f_x(0,0)$ and $f_y(0,0)$ exist.

$$f(x,y)=\frac{x^2y^2}{x^3+y^3}$$

$$f_x(0,0)=\lim_{h\to0}\frac{(h,0)-f(0,0)}{h}$$

$$=\lim_{h\to0}\frac{\frac{(h^2)(0)}{h^3+0}-0}{h}$$

$$=\lim_{h\to0}\frac{\frac{0}{h^3}}{h}$$

$$=\quad 0$$

$$f_y(0,0)=\lim_{k\to0}\frac{(0,k)-f(0,0)}{k}$$

$$=\lim_{k\to0}\frac{\frac{(0)(k^2)}{0+k^3}-0}{k}$$

$$=\quad 0$$

(9) $$f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2} & ,(x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$$ **[1993]**

then show that f_x and f_y are not continuous at $(0,0)$.

here

$$f_x = \frac{(x^2+y^2)(3x^2)-(x^3-y^3)(2x)}{(x^2+y^2)^2}$$
$$= \frac{3x^4+3x^2y^2-2x^4+2xy^3}{x^4+2x^2y^2+y^4}$$
$$= \frac{x^4+3x^2y^2+2xy^3}{x^4+2x^2y^2+y^4}$$

Thus $f_x(x,y) = \begin{cases} \dfrac{x^4+3x^2y^2+2xy^3}{x^4+2x^2y^2+y^4} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$

$$\therefore \lim_{(x,y)\to(0,0)} f_x(x,y)$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4+3x^2y^2+2xy^3}{x^4+2x^2y^2+y^4}$$

taking $y = mx = \phi(x)$ then $\phi(x)$ is continuous and $\phi(0)=0$

$$\lim_{(x,y)\to(0,0)} \frac{x^4+3x^2y^2+2xy^3}{x^4+2x^2y^2+y^4}$$
$$= \lim_{x\to 0} \frac{x^4+3x^2m^2x^2+2xm^3x^3}{x^4+2x^2m^2x^2+m^4x^4}$$
$$= \lim_{x\to 0} \frac{x^4(1+3m^2+2m^3)}{x^4(1+2m^2+m^4)}$$
$$= \lim_{x\to 0} \frac{1+3m^2+2m^3}{1+2m^2+m^4}$$
$$= \frac{1+3m^2+2m^3}{1+2m^2+m^4}$$

Here, for different values of m, above limit is different.

So, by the theorem, does not exist .

So , f_x is not continuous at $(0,0)$.

Now,

$$f_y = \frac{(x^2+y^2)(-3y^2)-(x^3-y^3)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-3x^2y^2-3y^4-2x^3y+2y^4}{x^4+2x^2y^2+y^4}$$

$$= \frac{-3x^2y^2-2x^3y-y^4}{x^4+2x^2y^2+y^4}$$

Thus,

$$f_y(x,y) = \begin{cases} \dfrac{-3x^2y^2-2x^3y-y^4}{x^4+2x^2y^2+y^4} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} f_y(x,y) = \lim_{(x,y)\to(0,0)} \frac{-3x^2y^2-2x^3y-y^4}{x^4+2x^2y^2+y^4}$$

Taking $y = mx = \phi(x)$ then $\phi(x)$ is continuous and $\phi(0)=0$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{-3x^2y^2-2x^3y-y^4}{x^4+2x^2y^2+y^4}$$

$$= \lim_{x\to0} \frac{-3x^2m^2x^2-2x^3mx-m^4x^4}{x^4+2x^2m^2x^2+m^4x^4}$$

$$= \lim_{x\to0} \frac{x^4(-3m^2-2m-m^4)}{x^4(1+2m^2+m^4)}$$

$$= \lim_{x\to0} \frac{-3m^2-2m-m^4}{1+2m^2+m^4}$$

$$= \frac{-3m^2-2m-m^4}{1+2m^2+m^4}$$

Here for above values of m, above limit is different.

So , by the theorem $\lim_{(x,y)\to(0,0)} f_y(x,y)$ does not exist .

So, f_y is not continuous at $(0,0)$.

Partial derivatives of second order

<u>Definition</u> :-Suppose partial derivatives $f_x(x,y)$ and $f_y(x,y)$ of a function $f(x,y)$ exist. We define and denote the partial derivative of second order as follows,

$$f_{xx}(a,b)=\lim_{h\to 0}\frac{f_x(a+h,b)-f_x(a,b)}{h}$$

$$f_{xy}(a,b)=\lim_{k\to 0}\frac{f_x(a,b+k)-f_x(a,b)}{k}$$

$$f_{yx}(a,b)=\lim_{h\to 0}\frac{f_y(a+h,b)-f_y(a,b)}{h}$$

$$f_{yy}(a,b)=\lim_{k\to 0}\frac{f_y(a,b+k)-f_y(a,b)}{k}$$

$$f$$

$$f_x \qquad\qquad f_y$$

$$f_{xx} \quad f_{xy} \qquad\quad f_{yx} \quad f_{yy}$$

Example1:

If $f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2} & ,(x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$

Find $f_{xx}(0,0)$, $f_{xy}(0,0)$, $f_{yx}(0,0)$ and $f_{yy}(0,0)$

Solution:- We have

$$f_{xx}(0,0)=\lim_{h\to 0}\frac{f_x(h,0)-f_x(0,0)}{h} \qquad ....(1)$$

Now $f_x(a,b)=\lim_{h\to 0}\dfrac{f(a+h,b)-f(a,b)}{h}$ ....(2)

$$\Rightarrow f_x(h,0)=\lim_{h\to 0}\frac{f(2h,0)-f(h,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0 \qquad ....(3)$$

Also from (2),

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{0-0}{h} = (0) \qquad ....(4)$$

Putting the values of (3) and (4) in (1)

$$f_{xx}(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\rightarrow$ We have,

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k} \qquad ....(1)$$

Now, $f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$ ....(2)

$$\therefore f_x(0,k) = \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(h)(k)(h^2-k^2)}{h^2+k^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^3k-k^3}{h^2+k^2} - 0}{h}$$

$$= -k \qquad ....(3)$$

Also from (2),

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = (0) \qquad ......(4)$$

Putting the values of (3) and (4) in (1)

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = \lim_{k\to 0}(-1) = -1$$

We have,

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} \qquad ......(1)$$

Now, $f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}$ ......(2)

$$\therefore f_y(h,0) = \lim_{k\to 0} \frac{f(h,k)-f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{(h)(k)(h^2-k^2)}{h^2+k^2} - 0}{k}$$

$$= \lim_{k\to 0} \frac{\frac{h^3k - hk^3}{h^2+k^2}}{k}$$

$$= \lim_{k\to 0} \frac{h^3}{h^2}$$

$$= \lim_{k\to 0} h = h \qquad \text{...............(3)}$$

Also from(2), $f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0 \qquad \text{............(4)}$$

Putting values of (3) and (4)in (1)

$f_{yx}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$

Also $f_{yy}(0,0) = \lim_{k\to 0} \frac{f_y(0,k) - f_y(0,0)}{k}$ ......(5)

Now, $f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$ ......(6)

$\therefore f_y(0,k) = \lim_{k\to 0} \frac{f(0,2k) - f(0,k)}{k}$

$= \lim_{k\to 0} \frac{0-0}{k} = (0)$ ......(7)

Also from (6),

$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$

$= \lim_{k\to 0} \frac{0-0}{k} = 0$ ......(8)

Putting values of (7) and (8) in (5),

$f_{yy}(0,0) = \lim_{k\to 0} \frac{0-0}{k} = (0)$

Note:- This example can also be asked as follows

If $f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$

then show that $f_{xy}(0,0) \neq f_{yx}(0,0)$

example2 **[2012,2009,2007]**

If $\mu = f(r)$ where $r^2 = x^2 + y^2 + z^2$ then prove that

$$\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \mu}{\partial z^2} = f''(r) + \frac{2}{r} f'(r).$$

Solution: We have, $r^2 = x^2 + y^2 + z^2$

Taking partial derivative with respect to *x*,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \text{............(1)}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ ....(2) and

$$\frac{\partial r}{\partial z} = \frac{z}{r} \qquad \text{............(3)}$$

Now, $\mu = f(r)$

$$\therefore \frac{\partial \mu}{\partial x} = f'(r)\frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial \mu}{\partial x} = f'(r)\frac{x}{r}$$

$$\mapsto \frac{\partial^2 \mu}{\partial x^2} = f'(r)\left(\frac{r - x\frac{\partial r}{\partial x}}{r^2}\right) + \frac{x}{r} f''(r)\frac{\partial r}{\partial x}$$

$$= f'(r)\left(\frac{r - x\left(\frac{x}{r}\right)}{r^2}\right) + \frac{x}{r} f''(r)\frac{x}{r}$$

$$= f'(r)\left(\frac{r^2 - x^2}{r^3}\right) + \frac{x^2}{r^2} f''(r)$$

Similarly,

$$\frac{\partial^2 \mu}{\partial y^2} = f'(r)\left(\frac{r^2 - y^2}{r^3}\right) + \frac{y^2}{r^2} f''(r)$$

$$\frac{\partial^2 \mu}{\partial z^2} = f'(r)\left(\frac{r^2 - z^2}{r^3}\right) + \frac{z^2}{r^2} f''(r)$$

So, $\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \mu}{\partial z^2}$

$$= \frac{f'(r)}{r^3}\left(r^2 - x^2 + r^2 - y^2 + r^2 - z^2\right) + \frac{f''(r)}{r^2}(x^2 + y^2 + z^2)$$

$$= \frac{f'(r)}{r^3}\left(x^2 + y^2 + z^2 - x^2 + x^2 + y^2 + z^2 - y^2 + x^2 + y^2 + z^2 - z^2\right) + \frac{f''(r)}{r^2}(x^2 + y^2 + z^2)$$

$$= \frac{f'(r)}{r^3}\left[2(x^2 + y^2 + z^2)\right] + f''(r)$$

$$= f''(r) + \frac{2f'(r)}{r}$$

$$\therefore \frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \mu}{\partial z^2} = f''(r) + \frac{2}{r} f'(r).$$

Example 3

If $u = r^n$ and $r^2 = x^2 + y^2 + z^2$ then show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = n(n+1)r^{n-2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$

$$= \quad n\, r^{n-1} \frac{x}{r}$$

$$= \quad n\, x\, r^{n-2}$$

So, $\frac{\partial^2 u}{\partial x^2} = n\,[\, r^{n-2} + x\,(n-2)\, r^{n-3} \frac{x}{r} .\,]$

$$= n\,[\, r^{n-2} + x^2\,(n-2)\, r^{n-4}\,]$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = n\,[\, r^{n-2} + y^2\,(n-2)\, r^{n-4}\,]$

and $\frac{\partial^2 u}{\partial z^2} = n\,[\, r^{n-2} + z^2\,(n-2)\, r^{n-4}\,]$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$= n\,[r^{n-2} + x^2\,(n-2)\, r^{n-4} + r^{n-2} + y^2\,(n-2)\, r^{n-4} + r^{n-2} + z^2\,(n-2)\, r^{n-4}]$

$= n\,[3r^{n-2} + (n-2)\, r^{n-4}\,(x^2 + y^2 + z^2)]$

$= n\,[3r^{n-2} + (n-2)\, r^{n-4}\,.\, r^2]$

$= n\,[3r^{n-2} + (n-2)\, r^{n-2}\,]$

$= n\, r^{n-2}\,[3 + n - 2]$

$= n\,(n+1)\, r^{n-2}$

Example 4

If u = f (x, y) with $x = e^s$ and $y = e^t$ then prove that

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} = x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$$

Here u = f (x, y)

Also $x = e^s$, $y = e^t$

Now $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s}$

$$= \frac{\partial u}{\partial x} \,.\, e^s$$

$$\therefore \frac{\partial^2 u}{\partial s^2} = \frac{\partial u}{\partial x} \,.\, e^s + e^s \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial x}{\partial s}$$

$$= \frac{\partial u}{\partial x}\ x + e^s \frac{\partial^2 u}{\partial x^2} e^s$$

$$= \frac{\partial u}{\partial x}\ x + \frac{\partial^2 u}{\partial x^2} (e^s)^2$$

$$= \frac{\partial u}{\partial x}\ x + \frac{\partial^2 u}{\partial x^2} x^2$$

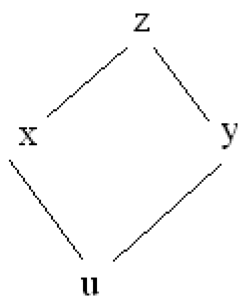
Similarly, $\frac{\partial u^2}{\partial t^2} = \frac{\partial u}{\partial y}\ y + \frac{\partial^2 u}{\partial y^2} y^2$

So, $\frac{\partial^2 u}{\partial s^2} + \frac{\partial u^2}{\partial t^2} = x \frac{\partial u}{\partial x} + x^2 \frac{\partial^2 u}{\partial x^2} + y \frac{\partial u}{\partial y} + y^2 \frac{\partial u^2}{\partial y^2}$

Example 5

z = f(x, y) with $x = e^{-u} + e^{v}$ and $y = e^{u} + e^{-v}$ then prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = y \frac{\partial z}{\partial y} - x \frac{\partial z}{\partial x}$$



Here, $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$

$$= -\frac{\partial z}{\partial x} e^{-u} + \frac{\partial z}{\partial y} e^{u} \qquad \text{.....(1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\ \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= +\frac{\partial z}{\partial x} e^{v} - \frac{\partial z}{\partial y} e^{-v} \qquad \text{.....(2)}$$

From (1) and (2), $\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}$

$$= \quad -\frac{\partial z}{\partial x} e^{-u} + \frac{\partial z}{\partial y} e^{u} - \frac{\partial z}{\partial x} e^{v} + \frac{\partial z}{\partial y} e^{-v}$$

$$= \quad \frac{e^{y} z}{\partial y}(e^{u} - e^{-v}) - \frac{\partial z}{\partial x}\left(e^{-u} + e^{v}\right) \frac{e^{y} z}{\partial y}(e^{u} - e^{-v}) - \frac{\partial z}{\partial x}\left(e^{-u} + e^{v}\right)$$

$$= \quad y\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial x}$$

➔ **Harmonic Function :-** If u = f (x, y) satisfies the equation

$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, then u is said to be a harmonic function.

Example 6 **:** If $u = \log(x^2 + y^2)$ then show that u is a harmonic function.

$u = \log(x^2 + y^2)$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2 + y^2\right)2 - 2x(2x)}{\left(x^2 + y^2\right)^2}$$

$$= \frac{2x^2 - 2y^2}{\left(x^2 + y^2\right)^2}$$

Similarly, $\dfrac{\partial u}{\partial y} = \dfrac{2y}{\left(x^2 + y^2\right)}$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2 - 2y^2}{\left(x^2 + y^2\right)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{\left(x^2 + y^2\right)^2} = 0$$

$\therefore$ u = log(x^2 + y^2) is a harmonic function.

- Practice Sums :

(1) If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$, then prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -9\left(x + y + z\right)^{-2}.$$

[hint : $= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

So, find $\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$ first. Using

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$,

we get $\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \frac{3}{x + y + z}$.]

(2) Show that $u = \log\left(x^2 + y^2\right)$ is a harmonic function of x and y.

(3) **[2008]** If $x = r\cos\theta\ ,\ \ y = r\sin\theta$ and u = f (x , y)

Then prove that $\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

[Hint : Using chain rule, $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} cos\theta + \frac{\partial u}{\partial y}\sin\theta$. Then $\frac{\partial^2 u}{\partial r^2} = cos^2\theta\ \frac{\partial^2 u}{\partial x^2} + sin^2\theta\ \frac{\partial^2 u}{\partial y^2}$.

Similarly, find $\frac{\partial^2 u}{\partial \theta^2}$. Then prove : RHS = …… = LHS.]

(4) **[2007]** If u = f (r) , $r^2 = x^2 + y^2 + z^2$, then prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$$

(5) If f(x,y) = c is an implicit function, then find

$$\frac{dy}{dx} \quad and \quad \frac{d^2 y}{dx^2}.$$

[hint : f(x , y) = c

$$\Rightarrow \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow f_x + f_y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}$$]

(6) **[2010]** If F (x, y, u, v) $= x^3 + y^3 + u^3 + 2v^3 - 5 = 0$

$$and\ G(x, y, u, v) = 2x^3 - y^3 + 3u^3 - v^3 - 7 = 0,$$

Find $\frac{\partial u}{\partial x}, \frac{\partial x}{\partial u}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 x}{\partial u^2},$

[Hint : $\frac{\partial u}{\partial x} = \frac{-\begin{vmatrix} Fx & Fv \\ Gx & Gv \end{vmatrix}}{\begin{vmatrix} Fu & Fv \\ Gu & Gv \end{vmatrix}}$ and

$$\frac{\partial x}{\partial u} = \frac{-\begin{vmatrix} Fu & Fy \\ Gu & Gy \end{vmatrix}}{\begin{vmatrix} Fx & Fy \\ Gx & Gy \end{vmatrix}}$$

Unit 3

Directional derivatives

Let f be a real valued function, which is well defined on some set S $\subset R^n$. If for $\bar{x}$ and $\bar{x}$ + hū $\in$ S (h $\neq$ 0), $\lim_{h\to 0} \frac{f(\bar{x}+h\hat{u})-f(\bar{x})}{h}$ exits, then it is said to be a directional derivative of function f($\bar{x}$) at point $\bar{x}$ in the direction of a unit vector ū. It is denoted by Dū f($\bar{x}$).

Examples:-

1) Find the directional derivative of f(x, y, z) = $2x^2 - y^2 + z^2$ at point (1, 2, 3) in the direction from point (1, 2, 3) to (3, 5, 0). **[2011]**

Here f(x,y,z) = $2x^2 - y^2 + z^2$

So, $f_x = 4x \Rightarrow (f_x)(1, 2, 3) = 4$

$f_y = -2y \Rightarrow (f_y)(1, 2, 3) = -4$

$f_z = 2z \Rightarrow (f_z)(1, 2, 3) = 6$

Also, ū = (3-1, 5-2 , 0-3)

= (2, 3, -3)

$\therefore$ |ū| = $\sqrt{4+9+9}$ = $\sqrt{22}$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \left(\frac{2}{\sqrt{22}}, \frac{3}{\sqrt{22}}, \frac{-3}{\sqrt{22}}\right)$$

Hence, the required directional derivative is ,

$= (f_x, f_y, f_z)_{(1,2,3)} \cdot \hat{u}$

$$= [4, -4, 6] \left(\frac{2}{\sqrt{22}}, \frac{3}{\sqrt{22}}, \frac{-3}{\sqrt{22}}\right)$$

$$= \frac{8}{\sqrt{22}} - \frac{12}{\sqrt{22}} - \frac{18}{\sqrt{22}}$$

$= \dfrac{-22}{\sqrt{22}} = -\sqrt{22}$

2) Find the directional derivative of f(x, y, z) = xy + yz + zx at point (1, 0, 0) in the direction of vector (1, 1, 1).

$f_x = y + z \Rightarrow (f_x)_{(1, 0, 0)} = 0$

$f_y = x + z \Rightarrow (f_y)_{(1, 0, 0)} = 1$

$f_z = y + x \Rightarrow (f_z)_{(1, 0, 0)} = 1$

Also, $\bar{u} = (1, 1, 1)$

$|\bar{u}| = \sqrt{1+1+1} = \sqrt{3}$

$\therefore \hat{u} = \dfrac{\bar{u}}{|u|} = \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

Hence the required directional derivative is given by,

$= (f_x, f_y, f_z)_{(1, 0, 0)} \cdot \hat{u}$

$= (0, 1, 1) \cdot \left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

$= 0 + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{3}} = \dfrac{2}{\sqrt{3}}$

3) If $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$ if $(x, y) \neq (0, 0)$

$= 0$ if $(x, y) = (0, 0)$

then find the directional derivative of f at point (0, 0) in the direction of a vector $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ We have,

$$D_{\hat{u}} f(\bar{x}) = \lim_{h \to 0} \frac{f(\bar{x} + h\hat{u}) - f(\bar{x})}{h}$$

$$D_{\left(1/\sqrt{2},\ 1/\sqrt{2}\right)} \text{ f}(0, 0) = \lim_{h\to 0} \frac{f\left[(0,0)+h\left(1/\sqrt{2}\ ,\ 1/\sqrt{2}\right)\right]-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{h}{\sqrt{2}},\frac{h}{\sqrt{2}}\right)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\left(h/\sqrt{2}\right)\left(h^2/2\right) \ / \ \left(h^2/2\right)+\left(h^4/4\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(h^3/2\sqrt{2}\right)\left(4/2h^2+h^4\right)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2}h^3}{h\left[h^2\left(2+h^2\right)\right]}$$

$$= \lim_{h\to 0} \frac{\sqrt{2}}{2+h^2}$$

$$= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

DIFFERENTIATION

Definition :

Let f : S $\to$ R be a real function well defined on a non empty open subset S of R^2.

Let (x, y) $\in$ S. Then f is said to be differentiable at point (x, y) if for any point (x+h , y+k) in the nbhd. of (x, y),

f(x+h , y+k) = f(x , y) + Ah + Bk + $\varepsilon\rho$,

Where (1) A and B are independent of h and k and

(2) $\rho = \sqrt{h^2 + k^2}$ with $\rho \to 0 \Rightarrow \varepsilon \to 0$.

Note : The constants A and B mentioned in the above definition are actually A = f_x and B = f_y.

Hence, the above expression may be expressed as :

$$f(x+h , y+k) = f(x , y) + f_x h + f_y k + \varepsilon\rho.$$

➔ **Result :-**

Let z = f(x, y) be defined on an open set S $\subset$ R^2. If f is differentiable at point (a, b), then it is continuous at point (a, b).

Proof :-

- Given :- f is differentiable at point (a, b).
- To Prove :- f is continuous at point (a, b).

i.e. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$

Proof :- f is differentiable at point (a, b).

So by the definition of a differentiable,

$\therefore$ f (a + h, b +k) = f (a, b) + Ah + Bk + $\varepsilon\rho$

where (1) A and B are independent of h and k

and (2) $\rho = \sqrt{h^2 + k^2}$ with $\rho \to 0 \Rightarrow \varepsilon \to 0$

$$\lim_{(h,k)\to(0,0)} f(a + h, b + k) = \lim_{(h,k)\to(0,0)} (f(a, b) + Ah + Bk + \varepsilon\rho)$$

$$\therefore \lim_{(h,k)\to(0,0)} f(a + h, b + k) = f(a, b)$$

Take x = a + h, y = b +k. Then $h \to 0 \Rightarrow x \to a$ and $k \to 0 \Rightarrow y \to b$.

Thus, $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

$\therefore$ f is continuous at point (a, b)

➔ **Result :-**

If a function $z = f(x, y)$ has continuous partial derivatives in its domain and if the functions.

$\phi : t \to x = \phi(t)$ and

$\psi : t \to y = \psi(t)$ have continuous derivatives in their domain [a, b],

then $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Proof :-

Let $t, t + \delta t \in [a, b]$, where δt is an increment of t. this increments in t produces increments in x and y which will produce increment in z.

let δx, δy and δz be the increments in x, y and z respectively.

Thus, $\delta z = f(x + \delta x, y + \delta y) - f(x, y)$

$= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y) \quad \ldots (1)$

Now, since f_x exists, by Lagrange's mean value theorem in two variables,

$f(x + \delta x, y + \delta y) - f(x, y + \delta y) = \delta x.\ fx(x + \theta_1 \delta x, y + \delta y) \quad \ldots (2)$

where $0 < \theta_1 < 1$.

Similarly,

f_y exists and by Lagrange's mean value theorem,

$f(x, y + \delta y) - f(x, y) = \delta y\ .\ fy(x, y + \theta_2 \delta y) \quad \ldots (3)$

where $0 < \theta_2 < 1$

Putting the values from (2) and (3) in (1),

$\delta z = \delta x\ .\ fx(x + \theta_1 \delta x, y + \delta y) + \delta y.\ fy(x, y + \theta_2 \delta y)$

where $0 < \theta_1, \theta_2 < 1$

For $\delta t \neq 0$,

$$\therefore \frac{\delta z}{\delta t} = f_x(x + \theta_1 \delta x, y + \delta y)\frac{\delta x}{\delta t} + f_y(x, y + \theta_2 \delta y)\frac{\delta y}{\delta t}$$

Taking the $\lim_{\delta t \to 0}$,

$$\lim_{\delta t \to 0} \frac{\delta z}{\delta t} = \lim_{\delta t \to 0} f_x(x + \theta_1 \delta x, y + \delta y) \frac{\delta x}{\delta t}$$

$$+ \lim_{\delta t \to 0} f_y(x, y+\theta_2 \delta y) \frac{\delta y}{\delta t}$$

$$\frac{dz}{dt} = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} f_x(x + \theta_1 \delta x, y + \delta y) \frac{\delta x}{\delta t} +$$

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} f_y(x, y+\theta_2 \delta y) \frac{\delta y}{\delta t} \qquad \ldots (4)$$

Since f_x and f_y are continous,

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} f_x(x + \theta_1 \delta x, y + \delta y) = f_x(x, y) = \frac{\partial z}{\partial x}$$

and $$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} f_y(x, y+\theta_2 \delta y) = f_y(x, y) = \frac{\partial z}{\partial y}$$

Hence from (4),

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Note :- We know that the partial derivatives of second order f_{xy} and f_{yx} are not equal every time. However, under certain conditions, they become equal. Following two theorems give such conditions for the equality of f_{xy} and f_{yx}.

➔ **Young's theorem :-**

If f is a real function defined on non empty open set $S \subset R^2$ and if f_x and f_y exist and they are differentiable at point (x, y) then $f_{xy} = f_{yx}$ at point (x, y).

Proof :-

Let D = f(x + h, y + k) – f(x + h, y) – f(x, y + k) + f(x, y)

Let $\phi(x, y)$ = f(x, y + k) – f(x, y).

$\therefore\ \phi(x + h, y) = f(x + h, y + k) – f(x + h, y)$

$\therefore\ \phi(x + h, y) - \phi(x, y) = f(x + h, y + k) – f(x + h, y) – f(x, y +k)$

$+ f(x, y) = D$

$\therefore D = \phi (x + h, y) – \phi (x, y)$

Since f_x exists, ϕ_x also exists. So, by Lagrange's mean value theorem, we get,

$D = h\ \phi_x (x + \theta_1 h, y)$ . . . (1) where $0 < \theta_1 < 1$

Now $\phi(x, y) = f(x, y + k) – f(x, y)$

$\therefore \phi(x + \theta_1 h, y) = f(x + \theta_1 h, y + k) – f(x + \theta_1 h, y)$

$\therefore \phi_x(x + \theta_1 h, y) = f_x(x + \theta_1 h, y + k) – f_x(x + \theta_1 h, y)$

Putting this value in (1), we get,

$D = h [f_x(x + \theta_1 h, y + k) – f_x(x + \theta_1 h, y)]$ . . . (2)

$D = h [(f_x(x + \theta_1 h, y + k) – f_x(x, y)) – (f_x(x + \theta_1 h, y) – f_x(x, y))]$ . . . (3)

Only for under standing

If f is differentiable,

$f(a + h, b + k) – f(a, b) = f_x h + f_y k + \varepsilon \rho$.

Since f_x is differentiable in our case, we replace f by f_x, A by f_{xx}, B by f_{xy}, a by x, h by θ_1h, b by y, we get,

$f_x(x + \theta_1 h, y + k) – f_x(x, y) = f_{xx}\ \theta_1 h + f_{xy}\ k + \varepsilon_1 \rho_1$

Note that in the definition, A's value is f_x. But in our case, f is replaced by f_x so f_x will be replaced by $(f_x)_x = f_{xx}$. Similarly, f_y will be replaced by $(f_x)_y = f_{xy}$.

Since fx is differentiable, by the definition of differentiability,

$f_x (x + \theta_1 h, y + k) – f_x (x, y) = f_{xx}\theta_1 h + f_{xy}k+ \varepsilon_1 \rho_1$ . . . (4)

Where, $\rho_1 = \sqrt{(\theta_1 h)^2 + k^2}$ and $\rho_1 \to 0 \Rightarrow \varepsilon_1 \to 0$

Similarly, we have,

$f_x(x + \theta_1 h, y) – f_x(x, y) = f_{xx}\theta_1 h + f_{xy}0 + \varepsilon_2 \rho_2$ . . . (5)

Where, $\rho_2 = \sqrt{(\theta_1 h)^2} = \theta_1 h$ and $\rho_2 \to 0 \Rightarrow \varepsilon_2 \to 0$

Putting values from (4) and (5) in (3),

$\therefore$ D = h [$f_{xx}\theta_1 h + f_{xy} k + \varepsilon_1 \rho_1 - f_{xx}\theta_1 h - \varepsilon_2 \rho_2$]

$\therefore$ D = h [$f_{xy} k + \varepsilon_1 \rho_1 - \varepsilon_2 \rho_2$]

$$\therefore D = h\left(f_{xy} k + \frac{\varepsilon_1\rho_1 - \varepsilon_2\rho_2}{\sqrt{h^2+k^2}}\sqrt{h^2+k^2} \right)$$

Taking $\varepsilon = \dfrac{\varepsilon_1\rho_1 - \varepsilon_2\rho_2}{\sqrt{h^2+k^2}}$

D = h [$f_{xy} k + \varepsilon \rho$], where $\rho = \sqrt{h^2+k^2}$

$\varepsilon = \varepsilon_1 \rho_1 - \varepsilon_2 \rho_2$ and $\rho \to 0 \Rightarrow \varepsilon \to 0$

$\therefore$ D = h kf_{xy} + h $\varepsilon \rho$

$$\therefore \frac{D}{hk} = f_{xy} + \frac{\varepsilon\rho}{k}$$

$$\lim_{\substack{h\to o \\ k\to o}} \frac{D}{hk} = \lim_{\substack{h\to o \\ k\to o}} \left(f_{xy} + \frac{\varepsilon\rho}{k} \right)$$

$$\therefore \lim_{\substack{h\to o \\ k\to o}} \frac{D}{hk} = f_{xy} \quad \ldots (6)$$

Similarly, if we take,

$\Psi(x, y) = f(x + h, y) - f(x, y)$ then we get,

$$\lim_{\substack{h\to o \\ k\to o}} \frac{D}{hk} = f_{yx} \quad \ldots (7)$$

From (6) and (7),

$f_{xy} = f_{yx}$

➔ Schwarz theorem : -

Statement**: -** If f is a real function defined on non empty open set S $\subset R^2$ and if f_x, f_y and f_{xy} exist and are continuous in the neighborhood of point (x, y) then $f_{xy} = f_{yx}$ at point (x, y).

Proof :-

Let D = f(x + h, y + k) – f(x + h, y) – f(x, y + k) + f(x, y)

Let ϕ(x, y) = f(x, y + k) – f(x, y).

$\therefore$ ϕ(x + h, y) = f(x + h, y + k) – f(x + h, y)

$\therefore$ ϕ(x + h, y) - ϕ(x, y) = f(x + h, y + k) – f(x + h, y) – f(x, y +k)

+ f(x, y).

= D

$\therefore$ D = ϕ (x + h, y) – ϕ (x, y)

Since f_x exists, ϕ_x also exists.

So, by Lagrange's mean value theorem, we get,

$D = h\,\phi_x(x + \theta_1 h, y)$ . . . (1) where $0 < \theta_1 < 1$

Now ϕ(x, y) = f(x, y + k) – f(x, y)

$\therefore \phi(x + \theta_1 h, y) = f(x + \theta_1 h, y + k) – f(x + \theta_1 h, y)$

$\therefore \phi_x(x + \theta_1 h, y) = f_x(x + \theta_1 h, y + k) – f_x(x + \theta_1 h, y)$

Putting this value in (1), we get,

$D = h\,[f_x(x + \theta_1 h, y + k) – f_x(x + \theta_1 h, y)]$ ……. (2)

Now f_{xy} exists . Hence applying L.M.V. again,

$$D = h\left[k\left(f_{xy}\left(x+\theta_1 h, y+\theta_2 k\right)\right)\right] \quad \text{………(3)}$$

where $0 < \theta_1 \theta_2 < 1$

Observe for the next part : lim A = B $\Rightarrow$ A $\to$ B $\Rightarrow$ A = B + $\in$

Also, f_{xy} is continuous, so

$$\lim_{\substack{h\to o\\ k\to o}} f_{xy}\left(x+\theta_1 h, y+\theta_2 k\right) = f_{xy}\left(x,y\right) = f_{xy}$$

$$\Rightarrow f_{xy}\left(x+\theta_1 h, y+\theta_2 k\right) = f_{xy} + \varepsilon$$

Where, $\varepsilon \to 0 \ \ as \ \ h \to 0,\ k \to 0$

So from (3),

$$D = hk\left(f_{xy} + \varepsilon \right) \qquad \ldots\ldots\ldots\ldots\ldots(4)$$

Putting the value of D from (1), we get,

$$f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y) = hk\left(f_{xy} + \varepsilon \right)$$

$$\left[f(x+h, y+k) - f(x+h, y) \right] - \left[f(x, y+k) - f(x, y) \right] = hk\left(f_{xy} + \varepsilon \right)$$

$$\Rightarrow \left[\frac{\left[f(x+h, y+k) - f(x+h, y) \right]}{k} \right] - \left[\frac{\left[f(x, y+k) - f(x, y) \right]}{k} \right] = h\left(f_{xy} + \varepsilon \right)$$

$$\Rightarrow \lim_{k \to 0} \left[\frac{\left[f(x+h, y+k) - f(x+h, y) \right]}{k} \right] - \lim_{k \to 0} \left[\frac{\left[f(x, y+k) - f(x, y) \right]}{k} \right] = \lim_{k \to 0} \left[h\left(f_{xy} + \varepsilon \right) \right]$$

$$\Rightarrow f_y(x+h, y) - f_y(x, y) = h\left(f_{xy} + \varepsilon' \right) \text{ where } \varepsilon' = \lim_{k \to 0} \varepsilon$$

$$\Rightarrow \frac{f_y(x+h, y) - f_y(x, y)}{h} = f_{xy} + \varepsilon'$$

$$\Rightarrow \lim_{h \to 0} \left[\frac{f_y(x+h, y) - f_y(x, y)}{h} \right] = \lim_{h \to 0} \left(f_{xy} + \varepsilon' \right)$$

$$f_{yx} = f_{xy} \qquad \left(\because \lim_{h \to 0} \varepsilon' = \lim_{h \to 0} \left(\lim_{k \to 0} \varepsilon \right) = 0 \right)$$

Unit 4

HOMOGENEOUS FUNCTIONS AND EULER'S THEOREM

Definition :- Homogenous function : Let f be a real function defined on some set $S \subset R^n$. Then f is said to be a homogeneous function of degree m, if $f(t(x_1, x_2, x_3, \ldots x_n)) = t^m.f(x_1, x_2, \ldots x_n)$

Thus,

f (x, y) is homogenous function of degree m, if

$f(t(x, y)) = t^m.f(x, y)$

i.e $f(tx, ty) = t^m.f(x, y)$

Examples :-

(1) Let $f(x, y) = x^5 + y^5$

$\therefore f(tx, ty) = t^5x^5 + t^5y^5$

$= t^5(x^5 + y^5)$

$= t^5 (f(x, y))$

Thus, $f(tx, ty) = t^5f(x, y)$

Hence, f is a homogenous function of degree 5.

(2) $f(x, y) = \dfrac{x^2 + y^2}{x + y}$ is homogenous function of degree 1.

(3) $f(x, y) = \dfrac{x + y}{x - y}$ is homogenous function of degree 0.

⇨ **Euler's theorem for homogenous function :-**

[2006 to 2012]

If $f(x_1, x_2, \ldots x_r)$ is a homogenous differentiable function of degree n, then,

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots + x_r \frac{\partial f}{\partial x_r} = n.f(x_1, x_2, \ldots, x_r).$$

Proof :- f is a homogenous function of degree n.

Therefore, $f(tx_1, tx_2,\ldots,tx_r) = t^n f(x_1, x_2,\ldots,x_r)$

Let $X_1 = tx_1,\ X_2 = tx_2, \ldots\ldots, X_r = tx_r$ . . ………… (1)

$\therefore f(X_1, X_2,\ldots,X_r) = t^n f(x_1, x_2,\ldots,x_r)$

Now differentiating with respect to t.

$$\therefore \frac{\partial f}{\partial X_1}\cdot\frac{\partial X_1}{\partial t} + \frac{\partial f}{\partial X_2}\cdot\frac{\partial X_2}{\partial t} + \ldots + \frac{\partial f}{\partial X_r}\cdot\frac{\partial X_r}{\partial t} = n\,t^{n-1} f(x_1, x_2,\ldots,x_r).$$

Now from (1) ,

$$x_1 \frac{\partial f}{\partial X_1} + x_2 \frac{\partial f}{\partial X_2} + \ldots + x_r \frac{\partial f}{\partial X_r} = n\,t^{n-1} f(x_1, x_2,\ldots,x_r).$$

Taking t = 1,

$X_1 = x_1,\ X_2 = x_2, \ldots, X_r = x_r.$ (from (1))

$$\therefore x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots + x_r \frac{\partial f}{\partial x_r} = n f(x_1, x_2,\ldots,x_r).$$

⇨ **Corollaries :-**

(1) If f(x, y) is a homogenous function of degree n, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$.

(2) If f(x, y) is homogenous with degree n and if second order partial derivatives of f exist, then

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1) f(x, y)$$

Proof :- f is a homogenous function of degree n. So, by Euler's theorem,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y) \quad \ldots (1)$$

Now differentiating partially with respect to x and y respectively, we get

$$x \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y \frac{\partial^2 f}{\partial x\partial y} = n \frac{\partial f}{\partial x} \quad \ldots (2)$$

$$x \frac{\partial^2 f}{\partial x \partial y} + y \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} = n \frac{\partial f}{\partial y} \quad \ldots (3)$$

Multiplying (2) by x, (3) by y and adding, we get,

$$x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} + x\,y \frac{\partial^2 f}{\partial x \partial y} + y\,x \frac{\partial^2 f}{\partial x \partial y} + x \frac{\partial f}{\partial y} + y \frac{\partial f}{\partial y} = n\,x \frac{\partial f}{\partial y} + n\,y \frac{\partial f}{\partial y}$$

Using $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, we get

$$x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) = n \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right)$$

$$x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + n f(x, y) = n^2 f(x, y) \quad [\text{from } \ldots.(1)]$$

$$\therefore\; x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n(n-1) f(x, y).$$

Examples :-

(1) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, then prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$ = Sin 2u. **[2010,2011,2012]**

Solution:-

Let $z = \frac{x^3 + y^3}{x - y}$ = f(x, y)

Then $z\left(t_x, t_y\right) = \frac{t^3 x^3 + t^3 y^3}{tx - ty}$

$$= \frac{t^3 (x^3 + y^3)}{t(x - y)}$$

$$= t^2 \left(\frac{x^3 + y^3}{x - y} \right)$$

So, z is a homogenous function of degree 2.

Hence, by Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \quad ...(1)$$

Now $u = \tan^{-1}z$

$\therefore z = \tan u$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial(\tan u)}{\partial u}\frac{\partial u}{\partial x}$$

$$= \sec^2 u \frac{\partial u}{\partial x}$$

Similarly, $\frac{\partial z}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$

Putting these values in (1)

$$\therefore x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \text{Sin } 2u \quad \text{(Proved)}$$

(2) If $u = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

<u>Solution :-</u>

Let u = $u = \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$ = f(x, y) then u (tx, ty) = $\dfrac{tx+ty}{\sqrt{t}\sqrt{x}-\sqrt{t}\sqrt{y}}$

$$= \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$$

∴ u is a homogenous function of degree o.

Hence by Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0(u)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

(3) If $u=\sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ = ½ tan u**. [2007,2009]**

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u.$$

<u>Solution :-</u>

Let $z=\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ = f (x, y)

Then z (tx , ty) = $\dfrac{tx+ty}{\sqrt{t}\sqrt{x}-\sqrt{t}\sqrt{y}}$

$$=t^{\frac{1}{2}}\frac{(x+y)}{\sqrt{x}+\sqrt{y}}$$

So, z is a homogenous function of degree ½.

Hence, by Euler's theorem,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{1}{2} z \quad \ldots (1)$$

Now $u = \sin^{-1} z \quad \Rightarrow \quad z = \sin u$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{\partial(\sin u)}{\partial u} \frac{\partial u}{\partial x} = \cos u \frac{\partial u}{\partial x}$$

And $\frac{\partial z}{\partial y} = \cos u \frac{\partial u}{\partial y}$

Putting these values in (1),

$$\therefore x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2} \sin u.$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u.$$

(4)Verify Euler's theorem for f(x, y) = $x^n \log\left(\frac{y}{x}\right)$

Here f is a homogenous function of degree n.

So, by Euler's theorem,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y)$$

Now, $\frac{\partial f}{\partial x} = x^n \frac{x}{y}\left(\frac{-y}{x^2}\right) + \log \frac{y}{x} \; n x^{n-1}$

$$= -\frac{x^n}{x} + n x^{n-1} \log \frac{y}{x}$$

$$= x^{n-1}\left[n \log\left(\frac{y}{x}\right) - 1\right]$$

$$\frac{\partial f}{\partial y} = x^n \frac{x}{y}.\frac{1}{x}$$

$$= \frac{x^n}{y}$$

$$\therefore x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = x\ x^{n-1}\left[n\log\left(\frac{y}{x}\right)-1\right] + y\ \frac{x^n}{y}$$

$$= x^n\ [n\log\frac{y}{x} - 1\ ] + x^n$$

$$= x^n\ [n\log\frac{y}{x} -1 + 1]$$

$$= nx^n \log\frac{y}{x}$$

$$= n\ f(x, y)$$

$$\therefore x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n\ f(x, y)$$

(5) If $u = \tan^{-1}\frac{x}{y}$, then prove that $x \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Solution :- Here $u (t x , t y) = u (x , y) = t^0 u (x , y)$

So, u is homogenous function of degree 0.

$\therefore$ By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0\ (u)$$

$$\therefore x \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Theorem:−2 *If* $u=\varnothing(\mathrm{H})$ *is a function of a* homogeneous *function.*
$H=f(x,y)$ *of* degree m where partial derivatives of second order exists, then

(1) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=m\frac{f(u)}{f'(u)}\ ,\ f'(u)\neq 0 \qquad =G(u)\ say$

(2) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=G(u)\ [G'(u)\ -1]\ \ where\ H=f(x,y)=f(u)=\varnothing^{-1}(u)$

★ *Verify Euler's theorem for following* hom*ogenous equations*:

1. $f(x,y)=\frac{x^2-y^2}{x^2+y^2}\ ,\ (x,y)\neq(0,0)$
2. $f(x,y)=\sin^{-1}(\frac{y}{x})\ ,\ x\neq 0$
3. $f(x,y)=\tan^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\ ,(x,y)\neq(0,0)$
4. $f(x,y)=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$

If $u=\sin\left(\sqrt{x}+\sqrt{y}\right)$, *then prove that*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}$$

✧ *If* $u=\frac{\sqrt{x}-\sqrt{y}}{\sqrt[3]{x}+\sqrt[3]{y}}$ *then show that* $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$

✧ *If* $u=\tan^{-1}\frac{x^3+y^3}{x+y}\ ,x+y\neq 0$, *then prove that*

1) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$ *and*

2) $x^2\frac{\partial u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}\ \ y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2\mathrm{u}$

✧ *If* $z=x^2\cos\frac{y}{x}+y^2\sin\frac{x}{y}$, *then prove that*

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z.$$

✧ *If* $\tan z=\tan^{-1}\frac{x^{2/3}+x^{1/3}y^{1/3}}{x^5+y^5}$ *, then prove that*

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{13}{6}\sin 2\mathrm{z}\ \text{and}$$

$$\mathrm{x}^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=\frac{13}{6}\sin 2z\left(\frac{13}{3}\cos 2z+1\right)$$

✧ *If* $u=\sin^{-1}\frac{x^2+y^2}{x+y}\ ,\ x+y\neq 0$, *then prove that*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$$

✧ *If* $u=\sin\left(\sqrt{x}\sqrt{y}\right)$, *then prove that*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\cdot\cos\left(\sqrt{x}+\sqrt{y}\right).$$

✧ *If* $u=\tan^{-1}\frac{x^3+y^3}{\sqrt{x^2+y^2}}$ *, then prove that*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$$

- If $z = \sin^{-1}\frac{\sqrt{x^2+y^2}}{xy}$, then prove that

  $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = \tan^3 z.$

- If $u = \operatorname{cosec}^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$, then prove that

  (1) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12}\tan u.$

  (2) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y^2} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}\left(13+\tan^2 u\right).$

- If $f(x,y) = \sqrt{x^2-xy} \Rightarrow x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy} = 0$

- If $z = x^n f\left(\frac{y}{x}\right) + y^n g\left(\frac{x}{y}\right)$, then prove that

  $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z.$

- If $z = x^2\tan^{-1}\left(\frac{y}{x}\right) + y^3\sin^{-1}\left(\frac{x}{y}\right) - x^4 y(\log y - \log x)$ then find the value of $x^2\left(\frac{\partial^2 z}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 z}{\partial x \partial y}\right) + y^2\left(\frac{\partial^2 z}{\partial x^2}\right)$

If $u = f(v)$, *where v is homogeneous function of x and y of* degree n , then prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nvf'(v).$

- If $z = x\sin^{-1}\frac{x}{y} + y\cos^{-1}\frac{y}{x}$, then prove that

  $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$

- If $u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$, then prove that

  $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u$

- If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, then prove that

  $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$

- If $u = e^{x^3+y^3}$, then prove that

  $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u\log u.$

- If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then prove that

  $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u.$

Best of luck